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Production planning and inventory allocation of a single-product assemble-to-order system with failure-prone machines

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ABSTRACT

We consider the optimal production and inventory allocation of a single-product assemble-to-order system with multiple demand classes and lost sales. Each component is replenished by a dedicated machine that is subjected to unpredictable breakdowns. We find that the machine state not only influences the production and allocation decisions on its own component but also influences the decisions on the other components. Specifically, the optimal component production policy is a base-stock policy with the base-stock level non-decreasing in the inventory levels of the other components and the states of the other machines. The optimal component allocation policy is a rationing policy with the rationing level non-increasing in the inventory levels of the other components, the states of the other machines, and its own machine state. We use an exponential distribution to approximate the distribution of the total processing times and propose two heuristic policies to address the production and allocation decisions. The importance of taking machine failures into consideration is revealed through computational experiments.

1. Introduction

Machine failure, which renders production uncertain and curtails production capacity, is recognized as one of the major issues that challenge the management of production systems, especially the assemble-to-order (ATO) system. The ATO strategy, a popular operations management strategy, is wildly used in practice and has received plentiful research attention. Song and Zipkin (2003), and Benjaafar and El Hafsi (2006) review the literature on this topic. In the ATO system, the manufacturer only keeps inventory at the component level and postpones product differentiation to the final stage of production. Such a strategy provides product diversity, while at the same time enables production to quickly respond to customer demand. Suppose that the components share the same demand process and demand is satisfied only if all the components are available, then the supply uncertainty of one component will affect the performance of the other components. In this situation, the influence of machine failures on the ATO system is significant.

An effective way to cope with replenishment uncertainty and capacity constraint is to deploy the demand differentiation strategy, which differentiates demand into different classes and offers different services to different demand classes. Since different demands have different values to the firm or they incur different penalty costs for lost sales or delays, it is not necessary to satisfy all the demands when production is capacitated. Demand differentiation can be implemented through the inventory allocation policy, which determines whether or not to satisfy the demand from a certain class based on the current system state. Therefore how to jointly manage production and inventory allocation in the ATO system with failure-prone machines and multiple demand classes is an interesting problem to explore. Addressing this problem in this paper, we derive the structural properties of the optimal production and inventory allocation policies with respect to two decision criteria, namely the expected total discounted cost over an infinite horizon and the average cost.

Managing an ATO system with failure-prone machines is a challenge in practice. For example, Solectron and Flextronics, two of the largest contract manufacturers have adopted the ATO strategy (Benjaafar and El Hafsi, 2006). Many manufacturing firms in China, especially those in the high-tech electronics industry, use such a strategy, too. In the manufacturing systems of such firms, some of the components are outsourced while the other components are produced in-house. If the outsourced components are delivered in time, then the replenishment of the produced components becomes the key factor that affects system performance. The system consisting of the produced components can be viewed as an ATO system with endogenous lead times, which is the system that we study here.

In recent years considerable research has been devoted to the modeling and analysis of decentralized and centralized ATO

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Table 1
Literature on the ATO system.

Allocation category	Allocation policy	Related literatures
Component-based allocation policy	FIFO	Song (1998), Song et al. (1999), Song and Yao (2002), Song (2002), Lu et al. (2003, 2005), Lu and Song (2005), Hoen et al. (2010)
	Priority	Mirchandani and Mishra (2002)
Product-based	NHB	Lu et al. (2009), Song and Zhao (2008)
Allocation policy	Optimal Control	Benjaafar and El Hafsi (2006)

systems. In a decentralized ATO system, the system is managed from the component perspective, i.e., the system is divided into several subsystems, which are managed separately. Then each subsystem is treated as a single-component inventory system with multiple demand classes. A large body of literature has studied the optimal production and inventory allocation of a subsystem with endogenous lead times, see Ha (1997a, 2000) for the lost sales model, and Ha (1997b), de Vericourt et al. (2002), and Gayon et al. (2009) for the backorder model.

In a centralized ATO system, the optimal policies for the subsystems are not necessarily optimal for the centralized system. The demand correlations among the components are taken into consideration. The literature on inventory allocation policies in the centralized ATO system can be broadly classified into two categories: the component-based allocation (CBA) policy, such as the first-in-first-out (FIFO) policy and the priority allocation policy, and the product-based allocation policy (PBA), such as the no-holdback allocation (NHB) policy (the modified first-in-first-out (MFIFO) policy belongs to the NHB policy). Under the PBA policy, the inventory allocation decision is made based on a component's own state, as well as the states of the other components. On the contrary, the CBA policy allocates inventory only based on a component's own state, regardless of the states of the other components. Table 1 presents a summary classification of the literature on allocation policies.

We mainly review the literature on the optimal control of an ATO system that is most related to our paper. Benjaafar and El Hafsi (2006) study the optimal control of an ATO system with multiple demand classes and endogenous lead times. Extending Ha's (1997a) work to the ATO system, they show that a dynamic control policy is optimal. They find that the optimal control policies for the system with lost sales have similar structural properties with respect to the expected total discounted cost criterion and the average cost criterion. They also consider the backorder case with a single demand.

There is an abundance of research on the single-component system with machine failures. Akella and Kumar (1986), Bielecki and Kumar (1988), and Sharifnia (1988) consider deterministic demand models, while Feng and Yan (2000) and Feng and Xiao (2002) study stochastic demand models. They show that the base-stock policy is optimal. They all consider the single-class demand model and do not include inventory allocation as a decision variable. Cheng et al. (accepted for publication) consider a make-to-stock system with multiple demand classes and failure-prone machines. They show that the optimal production policy is a state-dependent base-stock policy and the optimal rationing policy is a rationing policy with state-dependent rationing levels. Different from the above literature, Gao et al. (2010) study the performance evaluation of an ATO system with machine failures. We extend Cheng et al. (accepted for publication) to an ATO system, which is similar to the one considered in Benjaafar and El Hafsi (2006), but with failure-prone machines. By formulating the system as a Markov decision process, we work out the structural properties of the optimal control policy. Specifically, the optimal production policy for each component is a base-stock policy with state-dependent base-stock levels and the optimal allocation policy is a rationing policy with rationing levels depending on the system states.

The remainder of the paper is organized as follows: We introduce the basic model in Section 2. We present the structural properties of the optimal control policies with respect to two different decision criteria in Section 3. In Section 4 we propose two heuristic policies to facilitate policy implementation in practice. In Section 5 we present computational experiments to examine the performance of the heuristic policies and the influence of machine failures on system performance. We conclude the paper and suggest future research directions in Section 6.

2. Model description

Consider a single-product ATO system that supplies products to satisfy the demands from n different classes. The system consists of m different types of components. One unit of the final product requires one unit of each component (if the product requires more than one unit of a certain type of component, we can re-scale the unit of that component). The demand from class $i,j=1,2,\ldots,n$, arrives according to an independent Poisson process with a rate λ_i and requires one unit of the product. The demand is said to be satisfied only if none of the components is out of stock; otherwise the demand is lost and incurs a lost sale cost c_i, which varies from class to class (the demands with equal lost sale costs can be aggregated and treated as from the same class). Without loss of generality, we assume $c_1 > c_2 > \cdots > c_n$. The component $j,j=1,2,\ldots,m$, is replenished by its corresponding dedicated machine j. The processing time of component j is exponentially distributed with a production rate μ_j. Each machine is subjected to unpredictable breakdowns. We assume that machines failures are independent and time-dependent only. The up time of machine i follows an exponential distribution with a failure rate b_j. A down machine is sent to repair immediately and will resume its functional state after repair. The repair time of machine j follows an exponential distribution with a repair rate r_j.

Given the differences in the lost sale costs of different demand classes, it is generally not optimal to satisfy demands on the first-come-first-served (FCFS) basis regardless of their classes. Inventory rationing may be used to preserve inventory for demands with higher lost sale costs by rejecting those with lower lost sale costs. Inventory rationing has been shown to be an effective policy to save cost for systems with multiple demand classes. On the other hand, the production of a component is inevitably affected by the inventory levels of the other components because demand is satisfied only if all the components are available, so the stock out of one component affects the fulfillment of the demand. Hence the static base-stock policy may not be optimal.

We address the above problem by finding the optimal production and inventory allocation policies that jointly minimize the inventory-related cost with respect to two different decision criteria: the expected total discounted cost over an infinite horizon and the average cost. The production policy specifies whether or not to produce and which component to produce. The inventory allocation policy specifies whether or not to satisfy the demand from a certain class. To facilitate notation, let $X_i(t)$ be the inventory level of component i at time t, which cannot be negative, i.e., $X_i(t)\in Z^+$, and $M_i(t)$ be the state of machine i at time t. Each machine has only two states: 0 and 1, where 0 denotes that the machine is down and 1 denotes that the machine is functional. Then the system state at time t is $(X(t),M(t))$ with state space Ω, where $X(t)=(X_1(t),X_2(t),\cdots,X_m(t)),M(t)=(M_1(t),M_2(t),\cdots,M_m(t))$.

3. Optimal control

In this section we characterize the structural properties of the optimal control policies with respect to two different decision criteria.

3.1. The expected total discounted cost criterion

Let $J^\pi(X,M)$ denote the expected total discounted cost over an infinite horizon with a starting state (X,M) under policy π. Then $J^\pi(X,M)$ is given by

$$J^\pi(X,M)=\int_0^{+\infty} e^{-\beta t}\left[h(X(t))dt+\sum_{j=1}^{n} c_j dN_j^\pi(t)\right], \tag{1}$$

where β is the discount factor, $h(X(t))=\sum_{i=1}^m h_i(X_i(t))$, $h_i(X_i(t))$ is an increasing convex function denoting the holding cost rate of component i, and $N_j^\pi(t)$ is the total number of class j demands that cannot be satisfied immediately from the on-hand inventory up to time t under policy π. A policy π^* is said to be the optimal control policy if it satisfies

$$J^{\pi^*}(X,M)=\min_\pi J^\pi(X,M). \tag{2}$$

To facilitate analysis, we define a set F that records the indices of the functional machines, i.e., if $i\in F$, then $M_i=1$. So F is a subset of $\Phi,\Phi=\{1,2,\ldots,m\}$, i.e., $F\subseteq\Phi$. $\overline{F}$ denotes the complementary of F. Following Lippman (1975), we re-scale the time unit so that $\beta+\sum_{i=1}^m(\mu_i+b_i+r_i)+\sum_{i=1}^n\lambda_j=1$. Then re-writing (1), we obtain the following optimality equation:

$$\begin{aligned}J^{\pi^*}(X,M)&=TJ^{\pi^*}(X,M)=h(X)+\sum_{l=1}\lambda_l T^l J^{\pi^*}(X,M)+\sum_{p\in F}\mu_p T_p J^{\pi^*}(X,M)\\&+\sum_{p\in F} b_p J^{\pi^*}(X,M-e_p)+\sum_{q\in\overline{F}} r_q J^{\pi^*}(X,M+e_q)\\&+\left[\sum_{q\in\overline{F}}(\mu_q+b_q)+\sum_{p\in F} r_p\right]J^{\pi^*}(X,M),\end{aligned} \tag{3}$$

for any $(X,M)\in\Omega$, where e_p, $p=1,2,\ldots,m$ is the pth unit vector of dimension m, i.e., $e_p=(0,\ldots,1,\ldots,0)$, e is an m-dimensional vector of ones, i.e., $e=1,1,\ldots,1$, and T, T_p and T^l are operators defined on the real-valued function $u(X,M)$ on the state space Ω. We have

$$T_p u(X,M)=\min\{u(X+e_p,M),u(X,M)\}, \tag{4}$$

$$T^l u(X,M)=\min\{c_l+u(X,M),H^l u(X,M)\}, \tag{5}$$

where

$$H^l u(X,M)=\begin{cases} c_l+u(X,M), & \Pi_{i=1}^m X_i=0,\\ u(X-e,M), & \text{otherwise}.\end{cases}$$

Obviously, T_p determines whether or not to produce component p while T^l determines whether or not to satisfy a demand from class l. The first term in the optimality equation (3) denotes the holding cost. The second to the fifth terms denote the expected total discounted cost from the next decision epoch to infinity with uniform transition probability. The last term is obtained by the uniformization procedure.

It is known that the policy that satisfies the optimality equation is the optimal control policy. From the optimality equation, we can find that it is optimal to satisfy a demand from class l if and only if $J^{\pi^*}(X,M)-J^{\pi^*}(X-e,M)\ge -c_l$ and at the same time all the components are available. When machine p is functional, it is optimal to produce component p if and only if $J^{\pi^*}(X+e_p,M)-J^{\pi^*}(X,M)\le 0$.

The structural properties of the optimal control policy are characterized through the optimality equation. To facilitate analysis, let (X_{-i},M_{-i}) denote the system state excluding the inventory level of component i and the state of machine i, i.e., $(X_{-i},M_{-i})=(X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_m,M_1,\ldots,M_{i-1},M_{i+1},\ldots,M_m)$. The structural properties of the optimal control policy are presented in the following proposition.

Proposition 1. The optimal cost function is $J^{\pi^*}(X,M)\in V$ for any $(X,M)\in\Omega$. The optimal control policy can be characterized as follows:

(1) *Optimal production policy*: The optimal production policy for component i is a base-stock policy with the state-dependent base-stock level $S_i^*(X_{-i},M_{-i})$, where $S_i^*(X_{-i},M_{-i})=\min\{X_i : D_i J^{\pi^*}(X,M)\ge 0 \mid M_i=1\}$. Furthermore, $S_i^*(X_{-i},M_{-i})$ satisfies some additional properties as follows:

(1.a) $$S_i^*(X_{-i},M_{-i})\le S_i^*((X+e_j)_{-i},M_{-i}),\quad i\ne j, \tag{6}$$

(1.b) $$S_i^*(X_{-i},M_{-i})\le S_i^*(X_{-i},(M+e_j)_{-i}),M_j=0,\quad i\ne j. \tag{7}$$

(2) *Optimal allocation policy*: The optimal component allocation policy is a rationing policy with state-dependent rationing levels $R_i^*(X_{-i},M)$, where $R_i^*(X_{-i},M)=(R_{i,1}^*(X_{-i},M),R_{i,2}^*(X_{-i},M),\cdots,R_{i,n}^*(X_{-i},M))$. $R_{i,j}^*(X_{-i},M)$ denotes component i's rationing level for the demand from class j, where $R_{i,l}^*(X_{-i},M)=\min\{X_i : D_e J^{\pi^*}(X-e,M)\ge -c_l \mid \Pi_{i=1}^m X_i\ne 0\}$. Furthermore, the rationing level has some additional properties as follows:

(2.a) $$R_{i,l}^*(X_{-i},M)\ge R_{i,l}^*((X+e_j)_{-i},M),\quad i\ne j, \tag{8}$$

(2.b) $$R_{i,l}^*(X_{-i},M)\ge R_{i,l}^*(X_{-i},M+e_j),M_j=0,\quad j=1,2,\ldots,m, \tag{9}$$

(2.c) $$1=R_{i,1}^*(X_{-i},M)\le R_{i,2}^*(X_{-i},M)\le\cdots\le R_{i,n}^*(X_{-i},M)\le S_i^*(X_{-i},M_{-i}). \tag{10}$$

Proof. The proof is given in the Appendix □

Proposition 1 shows that the optimal control policy is dynamic and state-dependent. It is optimal to produce component i if $X_i<S_i^*(X_{-i},M_{-i})$; otherwise do not produce. As for the optimal allocation policy, it is optimal to satisfy a demand from class l if $X_i\ge R_{i,l}^*(X_{-i},M)$ for any $i,i=1,2,\ldots,m$; otherwise reject it. (1a) and (1b) show that the optimal base-stock level $S_i^*(X_{-i},M_{-i})$ is non-decreasing in the inventory levels of the other components and the states of the other machines. This is, if it is optimal to produce a certain type of component in a given state, then it remains optimal to produce it when the inventory levels of the other components increase or the other machines complete their repairs. (2a) and (2b) show that the rationing level $R_{i,l}^*(X_{-i},M)$ is non-increasing in the inventory levels of the other components, the states of the other machines, and its own machine state. That is, if it is optimal to satisfy a demand from a certain class in a given state, then it remains optimal to satisfy that class of demand when the inventory levels of the other components increase or the machines complete their repairs. (2c) shows that it is always optimal to satisfy the demands from class 1 when all

the components are available and the rationing level for each demand is monotonous, i.e., the demand with a higher lost sale cost has a lower rationing level than that with a lower lost sale cost.

We numerically show the structure of the optimal control policy using a simple case with two components and three classes of demand. Fig. 1 shows the structure of the optimal production policy while Fig. 2 shows the structure of the optimal allocation policy. In Fig. 1, the optimal production policy for each component is characterized by two different curves associated with the state of the other machine: up or down. The four curves divide the state space into eight regions. We specify the production decision for each region in the figure. In Fig. 2 we present the optimal allocation policy in four sub-figures. These sub-figures show the optimal allocation decisions associated with four different machine states. The corresponding allocation decisions are also specified in each sub-figure. We see that the optimal allocation policy is specified by eight rationing curves, which are dynamically adjusted according to the system state.

3.2. The average cost criterion

In this section we discuss the structural properties of the optimal control policy with respect to the average cost criterion. Let $g^{\pi}(X,M)$ denote the average cost function under policy π with a starting state (X,M), i.e.,

$$g^{\pi}(X,M)=\lim_{\tau\to+\infty}\frac{1}{\tau}E\left[\int_0^{\tau}\left[h(X(t))\,dt+\sum_{j=1}^{n}c_j dN_j^{\pi}(t)\right]\right]. \tag{11}$$

A policy $\tilde{\pi}^*$ is said to be optimal under the average cost criterion if it satisfies

$$g^{\tilde{\pi}^*}(X,M)=\min_{\pi} g^{\pi}(X,M). \tag{12}$$

Proposition 2. The optimal control policy with respect to the average cost criterion possesses the same structural properties as those with respect to the expected total discounted cost criterion. The optimal production policy for each component is a base-stock policy with base-stock levels satisfying (1a) and (1b), while the optimal allocation policy is a rationing policy with rationing levels satisfying (2a)–(2c) in Proposition 1.

Proof. The proof is given in the Appendix. □

3.3. System with backorders

In this section we assume that the demands that cannot be satisfied from the on-hand inventory are backlogged. Exploring and characterizing the structural properties of the optimal control policy for the backorders case is much more complicated than that for the lost sales case because we need to keep track of the inventory levels of all the components, the machines states, as well as the backorder levels of different demand classes, which drastically increases the number of system states (Benjaafar and El Hafsi, 2006). So, following Benjaafar and El Hafsi (2006), we only analyze the system with a single demand class as an initial attempt to address this problem.

We assume that demand arrives according to a Poisson process with an arrival rate λ. The demand that cannot be fulfilled immediately from the on-hand inventory is backordered and incurs a backorder cost b per unit per unit time. Let $Y_i(t)$ denote the net inventory level of component i at time t and $(Y(t),M(t))$ denote the system state at time t, where $Y(t)=(Y_1(t),\ldots,Y_m(t))$.Let Θ be the system state space, so we have $\Theta=\{(Y,M):Y_i\in Z,M_i=0,1\}$, where Z is the set of integrals. The number of backorders in the system is denoted by $B(Y(t))=\max\{0,Y_1^-(t),\ldots,Y_m^-(t)\}$,where $Y_i^-(t)=\max\{0,-Y_i(t)\}$, $i=1,\ldots,m$. Then the on-hand inventory level of component i is given by $Y_i(t)+B(Y(t))$. Let $Q(Y(t))$ denote the

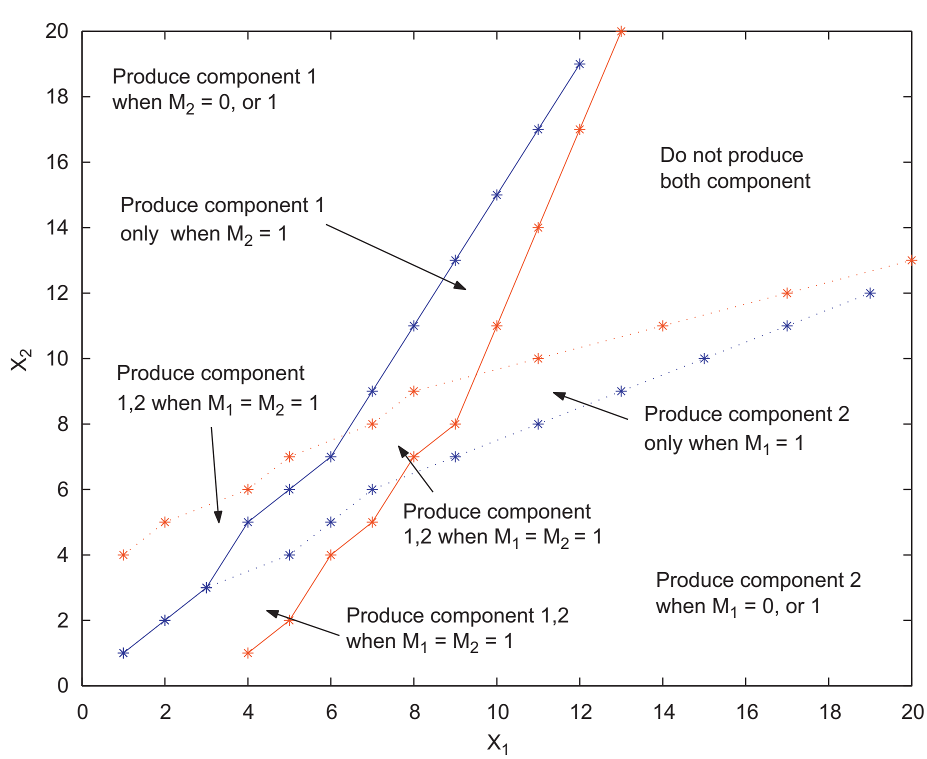

Fig. 1. Structure of optimal production policy for system with lost sales ($\lambda_1=\lambda_2=\lambda_3=1$, $\mu_1=\mu_2=2$, $b_1=b_2=0.1$, $r_1=r_2=0.2$, $c_1=160$, $c_2=80$, $c_3=40$, $h_1=h_2=1$, $\beta=0.0001$).

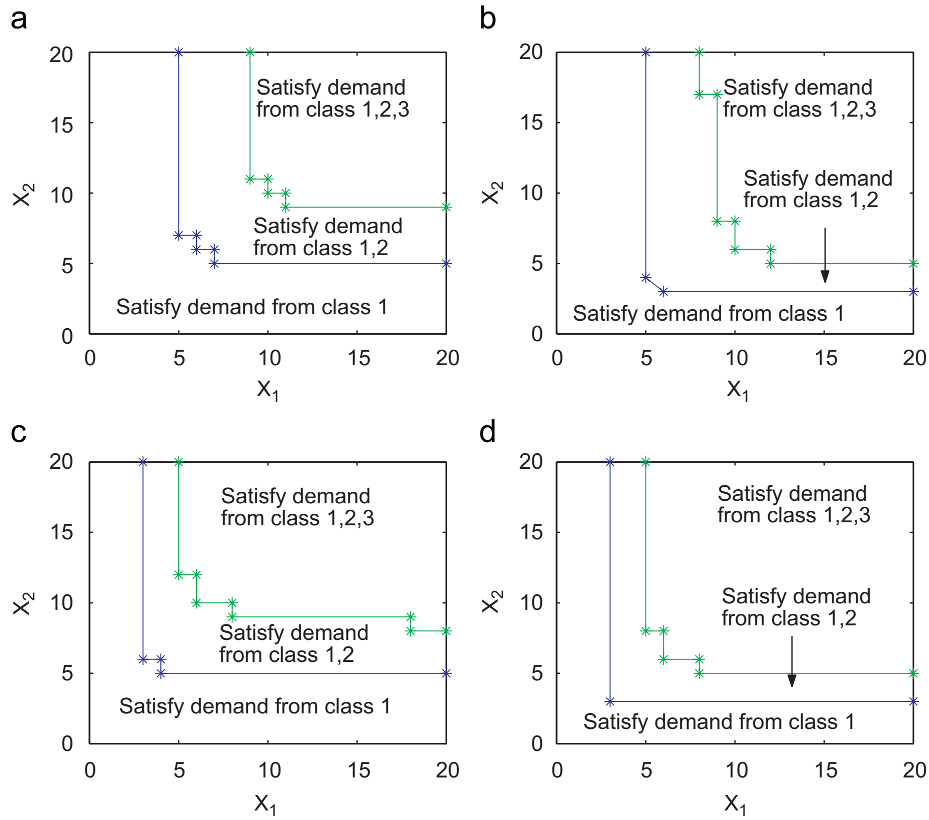

Fig. 2. Structure of optimal allocation policy for system with lost sales ($\lambda_1=\lambda_2=\lambda_3=1$, $\mu_1=\mu_2=2$, $b_1=b_2=0.1$, $r_1=r_2=0.2$, $c_1=160$, $c_2=80$, $c_3=40$, $h_1=h_2=1$, $\beta=0.0001$).

instantaneous cost at time t, i.e., $Q(Y(t))=\sum_{i=1}^{m}h_i(Y_i(t)+B(Y(t)))+bB(Y(t))$, where the first term is the holding cost while the second term is the backorder cost.

The expected total discounted cost function over an infinite horizon with a starting state (Y,M) under a feasible policy $\overline{\pi}$ is then given by

$$G^{\overline{\pi}}(Y,M)=\int_0^{+\infty}e^{-\beta t}Q(Y(t))\,dt. \tag{13}$$

A policy $\tilde{\pi}^*$is said to be the optimal control policy if it satisfies

$$G^{\tilde{\pi}^*}(Y,M)=\min_{\tilde{\pi}}G^{\tilde{\pi}}(Y,M).$$

To facilitate notation, we drop the superscript from $G^{\tilde{\pi}^*}(Y,M)$ used to denote the optimal cost function. Applying Lippman's transformation (1975), we re-scale the time unit so that $v'=\beta+\lambda+\mu_1+\mu_2+b_1+b_2+r_1+r_2$, and obtain the following equivalent optimality equation:

$$\begin{aligned}G(Y,M)=\tilde{T}G(Y,M)&=Q(Y)+\lambda G(Y-e,M)+\sum_{p\in F}\mu_p\tilde{T}_pG(Y,M)\\&+\sum_{p\in F}b_pG(Y,M-e_p)+\sum_{q\in\overline{F}}r_qG(Y,M+e_q)\\&+\left[\sum_{q\in\overline{F}}(\mu_q+b_q)+\sum_{p\in F}r_p\right]G(Y,M),\end{aligned} \tag{14}$$

where $\tilde{T}$ and $\tilde{T}_p$ are two operators defined on the state space Θ, $\tilde{T}_pG(Y,M)=\min\{G(Y+e_p,M),G(Y,M)\}$. Operator $\tilde{T}_p$ determines whether or not to produce component p.

The following proposition specifies the structural properties of the optimal control policy for the system with backorders.

Proposition 3. The optimal control policy for the backorder case retains the structural properties as those in the lost sales case. Specifically, the optimal production policy for component i is a dynamic base-stock policy with state-dependent base-stock level $\tilde{S}^*(Y_{-i},M_{-i})$, where $Y_{-i}=(Y_1,\ldots,Y_{i-1},Y_{i+1},\ldots,Y_m)$. Furthermore, the optimal policy satisfies the following additional properties:

(1) $\tilde{S}^*(Y_{-i},M_{-i})\le\tilde{S}^*((Y+e_j)_{-i},M_{-i}),\quad i\ne j,$
(2) $\tilde{S}^*(Y_{-i},M_{-i})\le\tilde{S}^*(Y_{-i},(M+e_j)_{-i}),\quad i\ne j,\quad M_j=0.$

Proof. The proof is given in the Appendix. □

The structure of the optimal control policy for the backorders case of an ATO system with two components is given in Fig. 3. In Fig. 3 we show that the optimal production policy is controlled by four state-dependent curves, which divide the state space into six regions. The optimal action for each region is specified in the figure.

4. Heuristic policies

The optimal control policy is dynamic and complex to implement in practice even for simple cases. In this section we propose two heuristic policies that have a relative simple structure and are easy to implement. The heuristic policies work on the basis of state space reduction by re-defining the processing time of each component. Noting that machine failures interrupt the production process and increases the total processing time, we re-define the total processing time of each component as consisting of the exact processing time and the total repair time, and denote it by Y_i. We use an exponential distribution with a mean $\tilde{\mu}_i^{-1}$ to approximate

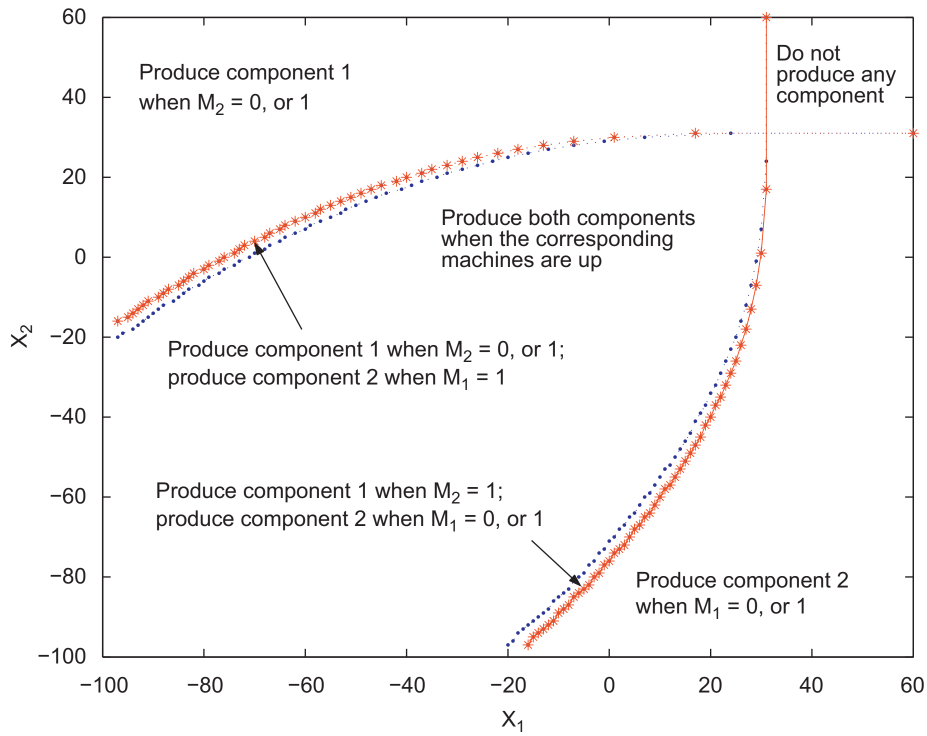

Fig. 3. Structure of optimal production policy for system with backorders ($\lambda=1$, $\mu_1=\mu_2=2$, $b_1=b_2=0.1$, $r_1=r_2=0.2$, $b=5$, $h_1=h_2=1$, $\beta=0.0001$).

the total processing time. In this section we propose two approximation methods to approximate $\tilde{\mu}_i$. Then the new system can be treated as the system with failure-free machines studied by Benjaafar and El Hafsi (2006). It is easy to see that, for any component, there exist 2^{m-1} machine-state-dependent base-stock levels and $(n-1)2^m$ machine-state-dependent rationing levels for that component under the optimal control policy. So the heuristic policies, which reduce the state dimension, can significantly simplify the computational effort to determine the optimal decisions. Furthermore, such heuristic policies can be used for either the expected total discounted cost or the average cost decision criterion. The Laplace–Stieltjes Transform (LST) of Y_i is given by (see Buzacott and Shanthikumar 1993)

$$\tilde{F}_{Y_i}(s)=\frac{\mu_i(r_i+s)}{s^2+s(r_i+b_i+\mu_i)+r_i\mu_i}. \tag{15}$$

We use two methods to approximate the total processing time. One is expectation approximation (EA), where $\tilde{\mu}_i^{-1}$ is approximated by the expectation of Y_i, denoted by $E(Y_i)$, i.e., $\tilde{\mu}_i^{-2}=E(Y_i)$. The other is variance approximation (VA), where $\tilde{\mu}_i^{-2}$ is approximated by the variance of Y_i, denoted by $D(Y_i)$, i.e., $\tilde{\mu}_i^{-2}=D(Y_i)$. We note that $E(Y_i)$ and $D(Y_i)$ can be calculated from the LST of Y_i. From (15), we have

$$E(Y_i)=-\tilde{F}'_{Y_i}(s)\big|_{s=0}=\frac{r_i+b_i}{r_i\mu_i}, \tag{16}$$

$$E(Y_i^2)=\tilde{F}''_{Y_i}(s)\big|_{s=0}=\frac{2(r_i+b_i)^2+2b\mu}{(r_i\mu_i)^2}, \tag{17}$$

$$D(Y_i)=E(Y_i^2)-[E(Y_i)]^2=\frac{(r_i+b_i)^2+2b\mu}{(r_i\mu_i)^2}. \tag{18}$$

Then the optimal values for the heuristic policy with respect to both decision criteria can be derived by substituting $\tilde{\mu}_i$ into μ_i in Benjaafar and El Hafsi's (2006) model.

5. Computational experiments

In this section we present computational results to examine the structures and assess the performance of the heuristic policies. The computational results highlight the significance of taking machine failures into consideration.

First we present some figures to show the structures of the heuristic policies for the systems with two components and three classes of demand with respect to the expected total discounted cost criterion (using the average cost criterion does not vary the structural properties of the optimal heuristic policies) and compare them with the optimal control policies. The optimal production and allocation control under the EA heuristic policy for the lost sales case are shown in Figs. 4–6. In Fig. 4 we present the optimal heuristic production decisions with the optimal production control curves in the background. We see that the heuristic production control divides the state space into four regions and the corresponding action for each region is specified in the figure. We note that the base-stock levels of both components under the heuristic policy are higher than those under the optimal control policy. This is possibly because the production rates of the components are under-estimated under the heuristic policy, which suggests the stocking of more inventory to cope with the uncertainty caused by machines failures. Fig. 5 presents the optimal heuristic allocation policy. We see that the rationing curves only divide the state space into three regions, which have a much simpler structure than that the optimal allocation policy. Fig. 6 presents the optimal heuristic allocation decisions with the optimal control policy in the background to facilitate

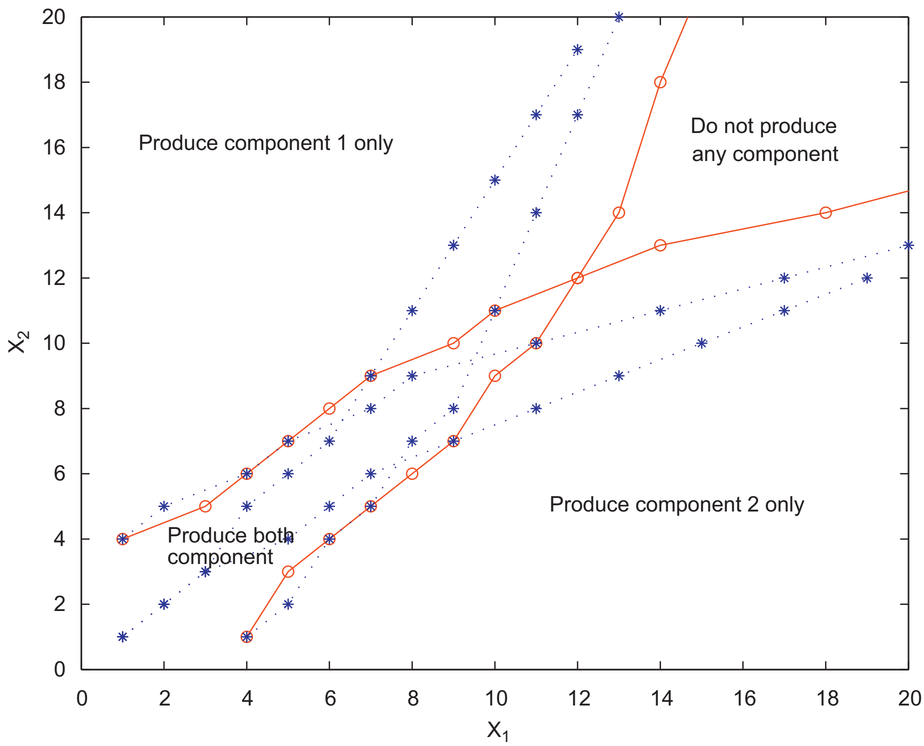

Fig. 4. Structure of EA heuristic production policy for system with lost sales. ($\lambda_1 = \lambda_2 = \lambda_3 = 1$, $\mu_1 = \mu_2 = 2$, $b_1 = b_2 = 0.1$, $r_1 = r_2 = 0.2$, $c_1 = 160$, $c_2 = 80$, $c_3 = 40$, $h_1 = h_2 = 1$, $\beta = 0.0001$).

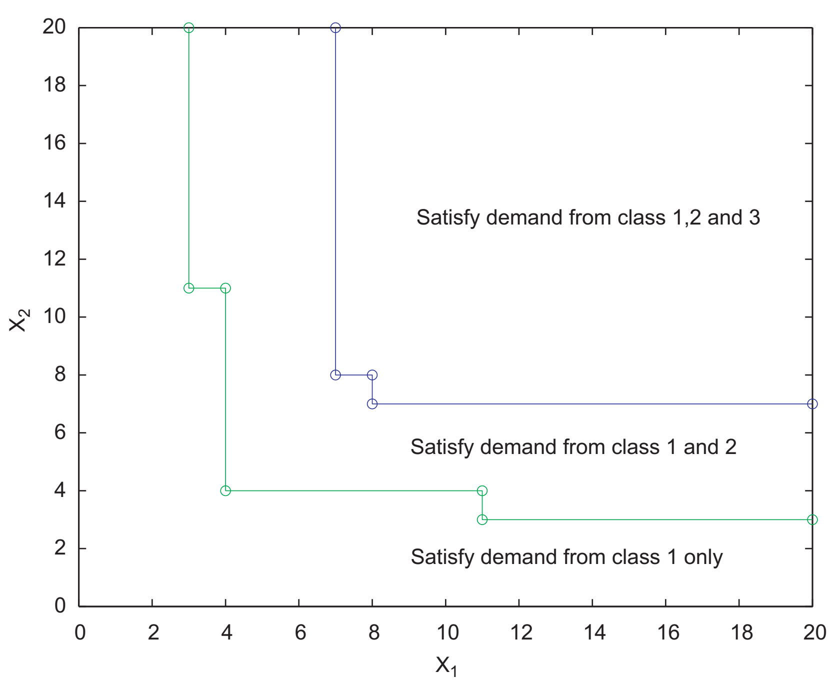

Fig. 5. Structure of EA heuristic allocation policy for system with lost sales ($\lambda_1 = \lambda_2 = \lambda_3 = 1$, $\mu_1 = \mu_2 = 2$, $b_1 = b_2 = 0.1$, $r_1 = r_2 = 0.2$, $c_1 = 160$, $c_2 = 80$, $c_3 = 40$, $h_1 = h_2 = 1$, $\beta = 0.0001$).

policy comparisons. We see that the rationing level for each demand class under the heuristic policy is not lower than that under the optimal control policy when the machines are both up and not higher than that when the machines are both down. Figs. 7 and 8 show the structure of the VA heuristic policy.

Figs. 9 and 10 present the optimal EA heuristic policies for the backorder case in different settings, one for the case with low failure rates and the other for the case with high failure rates. In Fig. 9 we see that the optimal heuristic production policy is characterized by two curves, which divide the state space into

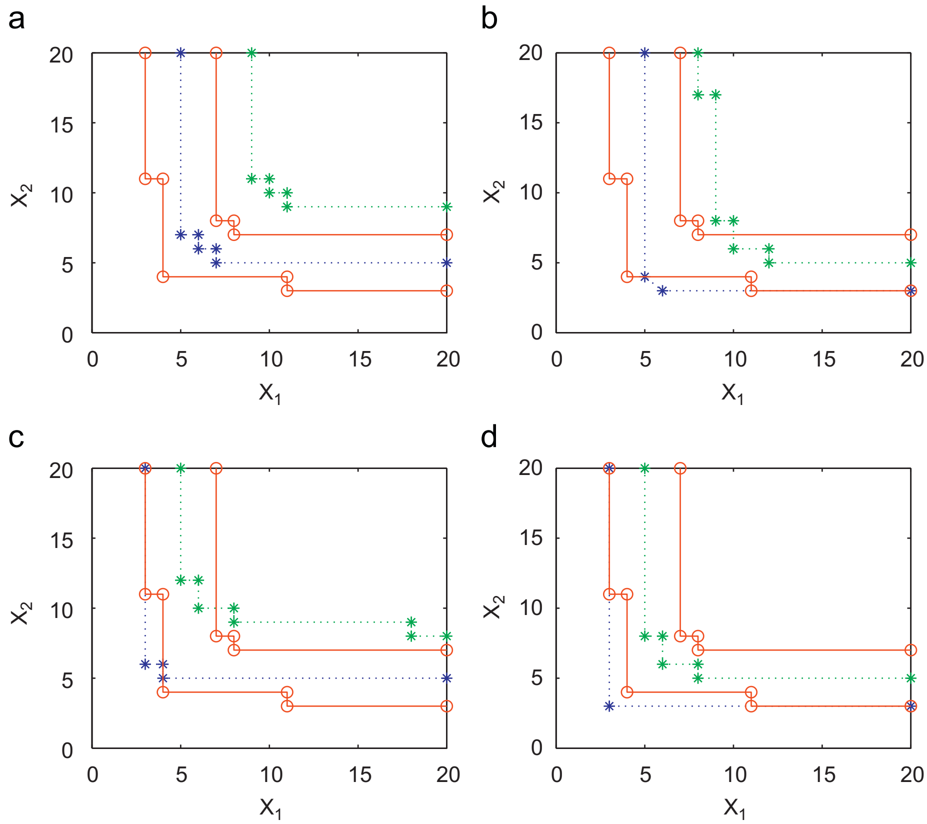

Fig. 6. EA heuristic allocation policy vs optimal allocation policy ($\lambda_1=\lambda_2=\lambda_3=1$, $\mu_1=\mu_2=2$, $b_1=b_2=0.1$, $r_1=r_2=0.2$, $c_1=160$, $c_2=80$, $c_3=40$, $h_1=h_2=1$, $\beta=0.0001$).

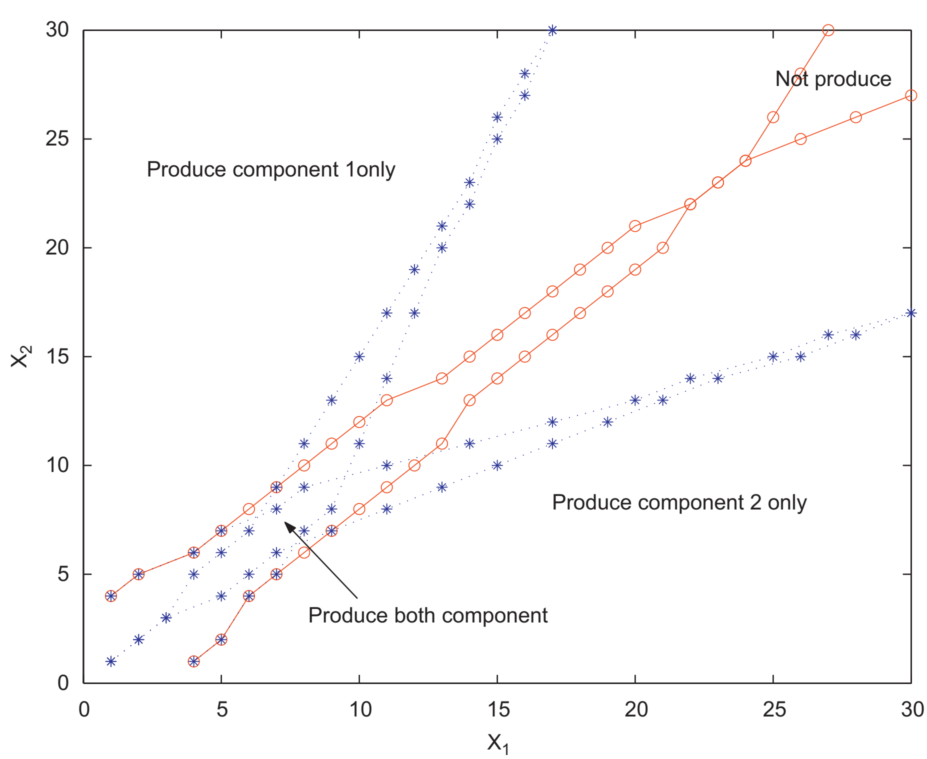

Fig. 7. Structure of VA heuristic production policy for system with lost sales ($\lambda_1=\lambda_2=\lambda_3=1$, $\mu_1=\mu_2=2$, $b_1=b_2=0.1$, $r_1=r_2=0.2$, $c_1=160$, $c_2=80$, $c_3=40$, $h_1=h_2=1$, $\beta=0.0001$).

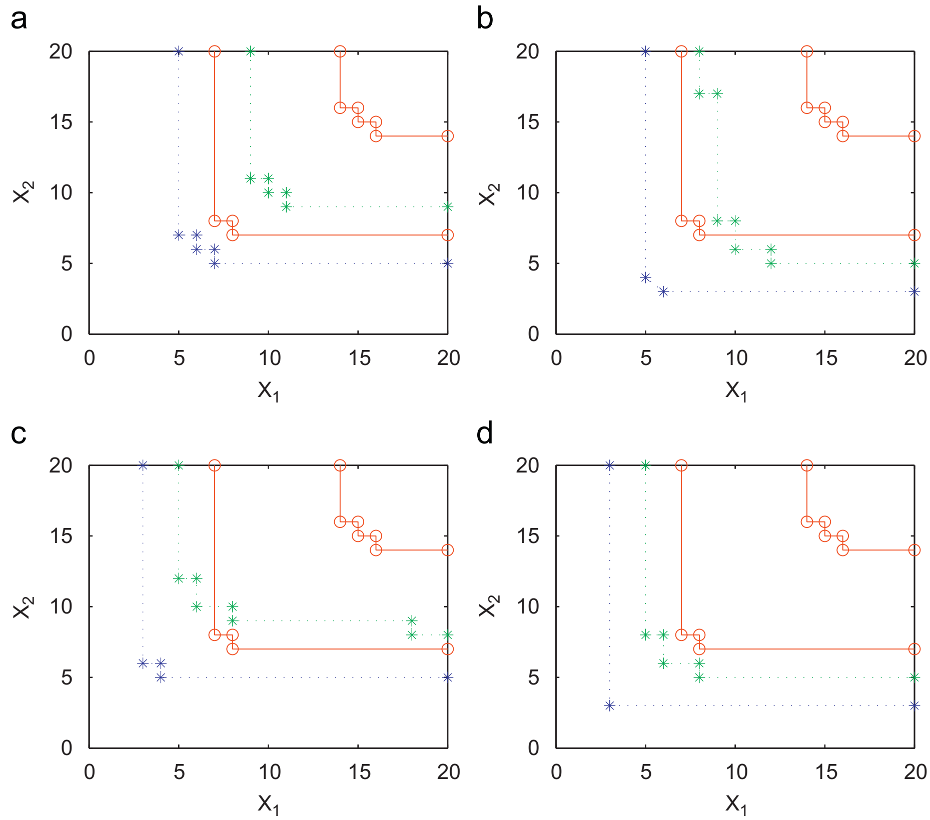

Fig. 8. VA heuristic allocation policy vs optimal allocation policy.($\lambda_1=\lambda_2=\lambda_3=1$, $\mu_1=\mu_2=2$, $b_1=b_2=0.1$, $r_1=r_2=0.2$,$c_1=160$, $c_2=80$, $c_3=40$, $h_1=h_2=1$, $\beta=0.0001$).

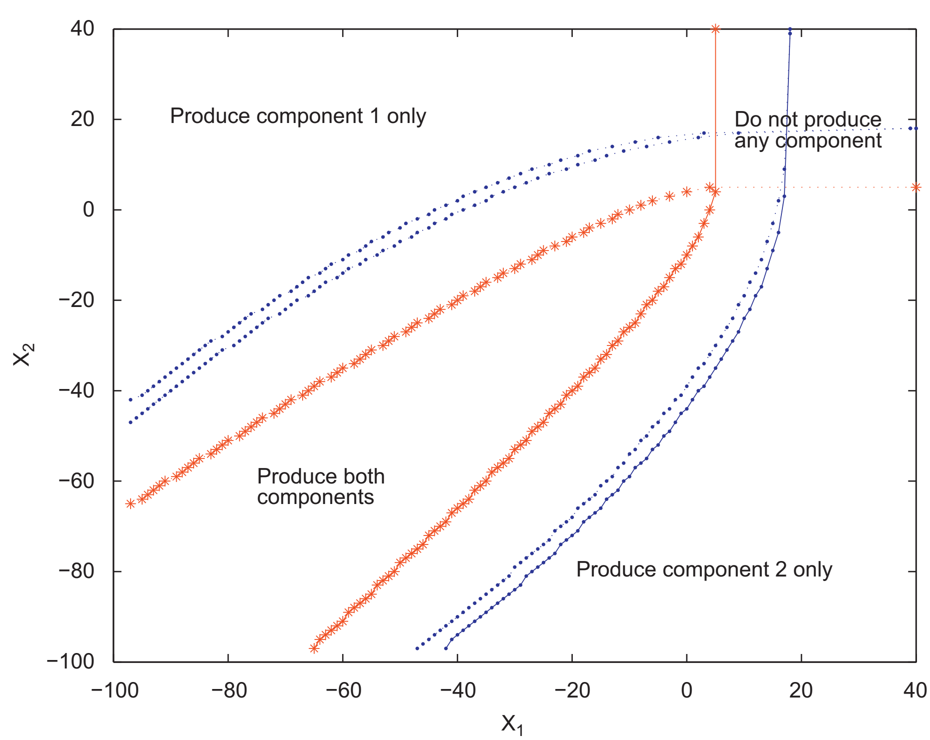

Fig. 9. Structure of EA heuristic allocation policy for system with backorders.($\lambda=1$, $\mu_1=\mu_2=2$, $b_1=b_2=0.1$, $r_1=r_2=0.2$,$b=5$, $h_1=h_2=1$, $\beta=0.0001$).

four regions. Each region has its own corresponding actions. We also note that the base-stock level for each component under the EA heuristic policy is lower than that under the optimal production policy. But it is not always the case, Fig. 10 shows that the base-stock level is not always lower than that the optimal base-stock level.

Now we provide computational experiments to show the effectiveness of exponential approximation and highlight the influence of

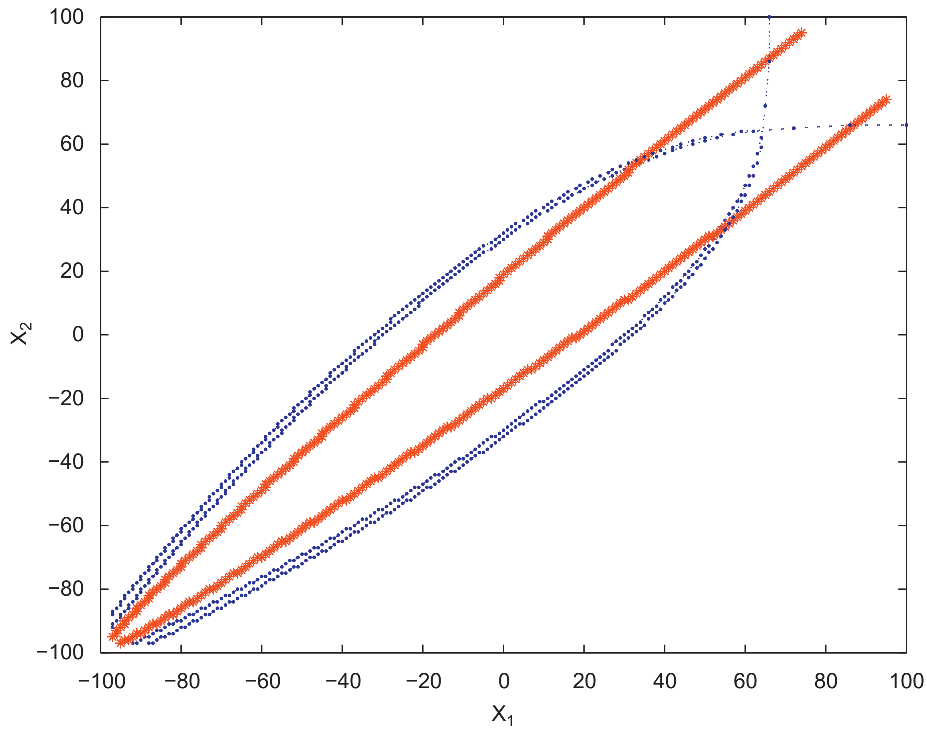

Fig. 10. Structure of EA heuristic allocation policy for systems with backorders ($\lambda=1$, $\mu_1=\mu_2=2$, $b_1=b_2=0.5$, $r_1=r_2=0.2$, $b=5$, $h_1=h_2=1$, $\beta=0.0001$).

machine failures. To facilitate comparisons, we only consider the average cost criterion because the average costs under the optimal control policy and a heuristic policy are both constant and independent of the starting state. The average cost under the optimal control policy can be computed by the value iteration algorithm. Before applying the value iteration algorithm, we first truncated the infinite countable state space to a finite state space, i.e., $[0,\bar{S}]$, where $\bar{S}$ is large enough to ensure that the optimal cost function is not sensitive to the truncated state space, then we redefined the transition rates. The optimal heuristic policy, which actually is the optimal control policy for the single-product ATO system with failure-free machines, can be obtained from Benjaafar and El Hafsi (2006). Then we applied the optimal heuristic policy to the original system and derived the average cost under this policy by using the value iteration algorithm. Details on the value iteration algorithm can be found in Puterman (1994).

We evaluate the effectiveness of a heuristic policy H, $H=EA$ or VA, by its relative cost difference from the optimal control policy, denoted by $R^H=(g^H-g/g)\times 100\%$, where g^H is the average cost of H and g is the optimal average cost. A small value of R^H indicates that the heuristic policy performs well compared with the optimal control policy. We ran two sets of computational experiments to examine the performance of the heuristic policies, which are listed in Table 2. From Table 2 we see that the ratio R^{EA} varies from 0.583 to 16.122 depending on the system settings. The heuristic policy works well when $\lambda_1+\lambda_2+\lambda_3>\mu_i$ because the average ratio of R^{EA} is about 1.500. But when $\lambda_1+\lambda_2+\lambda_3\le\mu_i$, the heuristic policy does not work so well with the ratio R^{EA} being around 10.000. In other words, as the production rate increases, the effectiveness of the approximate distribution of the actual processing time decreases. A plausible explanation is that the approximate distribution amplifies supply uncertainty as the production rate and failure rate increase, which prompts the system to store more than the necessary inventory to cope with the uncertainty. This can be induced from the seventh and eighth sets of numerical examples in Table 2. Note that the seventh set of numerical examples indicates that the ratio R^{EA} decreases with the lost sales cost of class 1 demand and increases with the holding cost of component 1. Specifically, R^{EA} reaches the highest point 16.122 when the holding cost is 5. From the ninth set of examples, we see that the ratio R^{EA} does not necessarily increase with the failure rates of both machines. This is possibly because the failure rates affect the system cost through the approximate distribution of the total processing time and the effectiveness of the approximate distribution does not necessarily increase with the failure rates. On the other hand, R^{VA} varies from 0.363 to 63.474. We also note that the ratio R^{EA} is smaller than the ratio R^{VA} most of the time, which implies that EA performs better than VA. Properly choosing an approximation method could be a key factor in the design of heuristic policies.

Table 2 also shows that the average cost increases with the failure rate of each machine, the lost sales cost of the demand with high priority, and the holding cost of each component, while decreases with the lost sales cost of the demand with low priority.

6. Conclusions

In this paper we consider the optimal production and inventory allocation of a single-product ATO system with failure-prone machines and multiple demand classes. We show that the optimal control policy depends on the component inventory levels as well as the machine states. In the lost sales case, the demand from the top class should always be satisfied if all the components are available. Rejecting the demands from the top class yields no benefit to the system. In view of the complexity of the optimal control policy, we propose two heuristic policies with relatively simple structure to facilitate policy implementation in

Table 2
Optimal control policy vs. heuristic policy w.r.t. average cost criterion.

λ_1	λ_2	λ_3	μ_1	μ_2	b_1	b_2	r_1	r_2	c_1	c_2	c_3	h_1	h_2	g^{π}	R^{HE}	R^{HV}
1	1	1	2	2	0.1	0.1	0.2	0.2	120	80	60	1	1	26.449	2.247	4.429
–	–	–	–	–	–	–	–	–	140	–	–	–	–	27.762	1.963	6.076
–	–	–	–	–	–	–	–	–	160	–	–	–	–	29.007	2.002	7.487
–	–	–	–	–	–	–	–	–	180	–	–	–	–	30.180	1.606	7.928
–	–	–	–	–	–	–	–	–	100	–	80	–	–	27.070	2.540	2.119
–	–	–	–	–	–	–	–	–	–	–	60	–	–	24.903	1.664	3.507
–	–	–	–	–	–	–	–	–	–	–	40	–	–	22.502	2.048	3.977
–	–	–	–	–	–	–	–	–	–	–	20	–	–	19.986	2.363	4.129
–	–	–	–	–	–	–	–	–	160	80	40	1	–	26.551	2.233	8.203
–	–	–	–	–	–	–	–	–	–	–	–	2	–	28.102	2.792	8.219
–	–	–	–	–	–	–	–	–	–	–	–	3	–	29.129	2.692	7.591
–	–	–	–	–	–	–	–	–	–	–	–	4	–	29.914	3.600	8.223
–	–	–	–	–	–	–	–	–	–	–	–	5	–	30.543	4.592	9.887
–	–	–	–	–	0.5	0.5	–	–	120	–	60	1	–	29.323	1.443	0.945
–	–	–	–	–	–	–	–	–	140	–	–	–	–	31.200	0.932	1.118
–	–	–	–	–	–	–	–	–	160	–	–	–	–	32.952	1.292	0.837
–	–	–	–	–	–	–	–	–	180	–	–	–	–	34.659	0.891	0.539
–	–	–	–	–	–	–	–	–	100	–	80	–	–	29.702	1.516	1.112
–	–	–	–	–	–	–	–	–	–	–	60	–	–	27.378	1.752	1.208
–	–	–	–	–	–	–	–	–	–	–	40	–	–	25.007	1.899	1.288
–	–	–	–	–	–	–	–	–	–	–	20	–	–	22.628	2.095	1.415
–	–	–	5	5	0.1	0.1	–	–	80	80	40	–	–	10.733	8.130	59.893
–	–	–	–	–	–	–	–	–	100	–	–	–	–	11.577	6.656	62.208
–	–	–	–	–	–	–	–	–	120	–	–	–	–	12.383	5.690	60.158
–	–	–	–	–	–	–	–	–	140	–	–	–	–	13.118	9.175	62.177
–	–	–	–	–	–	–	–	–	160	–	–	1	–	13.815	8.795	63.474
–	–	–	–	–	–	–	–	–	160	–	–	2	–	14.963	8.875	56.553
–	–	–	–	–	–	–	–	–	–	–	–	3	–	15.770	8.516	54.978
–	–	–	–	–	–	–	–	–	–	–	–	4	–	16.409	7.331	50.753
–	–	–	–	–	–	–	–	–	–	–	–	5	–	17.000	8.071	51.907
–	–	–	–	–	–	–	–	–	80	–	–	1	–	13.082	11.815	9.721
–	–	–	–	–	–	–	–	–	100	–	–	–	–	14.294	9.738	12.733
–	–	–	–	–	–	–	–	–	120	–	–	–	–	15.474	8.285	14.288
–	–	–	–	–	–	–	–	–	140	–	–	–	–	16.618	8.461	14.201
–	–	–	–	–	–	–	–	–	160	–	–	1	–	17.715	7.170	13.362
–	–	–	–	–	–	–	–	–	160	–	–	2	–	18.172	10.587	17.162
–	–	–	–	–	–	–	–	–	–	–	–	3	–	18.441	14.315	20.465
–	–	–	–	–	–	–	–	–	–	–	–	4	–	18.645	14.417	22.510
–	–	–	–	–	–	–	–	–	–	–	–	5	–	18.773	16.122	22.839
–	–	–	5	5	0.1	0.1	–	–	–	–	–	1	–	13.815	8.795	63.474
–	–	–	–	–	0.2	0.2	–	–	–	–	–	–	–	16.063	8.193	37.129
–	–	–	–	–	0.3	0.3	–	–	–	–	–	–	–	17.035	6.269	25.506
–	–	–	–	–	0.4	0.4	–	–	–	–	–	–	–	17.507	7.260	19.067
–	–	–	–	–	0.5	0.5	–	–	–	–	–	–	–	17.715	7.170	13.362

practice. The two heuristic policies differ in the method used to approximate the total processing time. We find that the performance of both heuristic policies depends on the system settings and EA outperforms VA most of the time.

Extending the model to the system with multiple demand classes and backorders, we see that the optimal control policy is much more complicated because both the production and inventory allocation decisions depend on the number of backlogs of each class of demands. The structural properties of such a backorder model are unknown. In this paper we assume that the assembly time is negligent. A new model should be formulated to explore the optimal control policy for case of non-negligent assembly time. A more general model is to study the optimal control of a multi-stage ATO system with failure-prone machines. Another potential research direction is to consider the case of multiple machine states. We consider that each machine has only two states: up and down. But in reality machines can transcend across various stages. The case of multiple machine states naturally raises the question of preventive maintenance. Exploring the optimal preventive maintenance policy for such an ATO system would be an interesting direction for future research.

Acknowledgments

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Appendix

Proof of Proposition 1

In order to specify these properties, we introduce a set of functions with certain properties and prove, following the same logic in deriving the optimality equation, that the operator defined possesses these properties. To simplify notation, we define the following difference operators:

$$D_i u(X,M) = u(X+e_i,M) - u(X,M),$$
$$D_e u(X,M) = u(X+e,M) - u(X,M),$$
$$D_{ij} u(X,M) = D_i u(X+e_j,M) - D_i u(X,M) = D_j u(X+e_i,M) - D_j u(X,M),$$

$D_{ie}u(X,M)=D_iu(X+e,M)-D_iu(X,M)=D_eu(X+e_i,M)-D_eu(X,M)$,
$\Delta_iu(X,M)=u(X,M)-u(X,M-e_i)$, where $M_i=1$.

We see that D is the difference operator on the inventory level while Δ is on the machine state. Let V be a set of real-valued functions defined on the state space Ω. If $u(X,M)\in V,(X,M)\in\Omega$, then $u(X,M)$ satisfies the following properties:

$P1:\ D_{ii}u(X,M)\geq 0$,
$P2:\ D_{ij}u(X,M)\leq 0,\quad i\neq j$,
$P3:\ D_{ie}u(X,M)\geq 0$,
$P4:\ D_iu(X,M)-D_iu(X,M-e_i)\geq 0$, for any $i\in\{p,M_p=1\}$,
$P5:\ D_ju(X,M)-D_ju(X,M-e_i)\leq 0$, for any $i\in\{p,M_p=1\}$, and $i\neq j$,
$P6:\ D_eu(X,M)-D_eu(X,M-e_i)\geq 0$, for any $i\in\{p,M_p=1\}$,
$P7:\ D_eu(X,M)\geq -c_1$.

These properties are used to characterize the structure properties of the optimal control policy. Properties P1–P3 are introduced to characterize the relationship with the inventory levels. Specifically, Property P1 indicates that $u(X,M)$ is convex in X_i, $i=1,\ldots,m$. The convex property is used to prove that the optimal production control has a threshold-type structure. Property P2 indicates that $u(X,M)$ is submodular, which is used to specify how the optimal action varies with the other state variances in X_{-i}. Property P3 is used to characterize how the optimal allocation policy varies with changes in X_i, $i=1,\ldots,m$. P4–P5 are used to characterize how the optimal production policy and inventory allocation policy vary with the states of the machines. P7 provides a bound on $D_eu(X,M)$.

Properties P4–P6 can be also expressed in the following manner:

$P4':\ \Delta_iu(X+e_i,M)-\Delta_iu(X,M)\geq 0$, for any $i\in\{p,M_p=1\}$,
$P5':\ \Delta_iu(X+e_j,M)-\Delta_iu(X,M)\leq 0$, for any $i\in\{p,M_p=1\}$, and $i\neq j$,
$P6':\ \Delta_iu(X+e,M)-\Delta_iu(X,M)\geq 0$, for any $i\in\{p,M_p=1\}$.

Lemma 1. If $u(X,M)\in V$, then $Tu(X,M)\in V$.

Proof. The proofs of properties P1, P2, P3, and P7 are similar to those in Benjaafar and El Hafsi (2006), so we omit them. We prove properties P4–P6 here.

Verification of property P4

We find that if we can prove that the following conditions hold

$SP1:\ T_iu(X+e_i,M)-T_iu(X,M)\geq u(X+e_i,M-e_i)-u(X,M-e_i)$,
$SP2:\ D_iT_pu(X,M)-D_iT_pu(X,M-e_i)\geq 0,\quad i,p\in F$, but $i\neq p$,
$SP3:\ D_iT^lu(X,M)-D_iT^lu(X,M-e_i)\geq 0$,

then $D_iu(X,M)-D_iu(X,M-e_i)\geq 0$ holds.

SP1: $A_1u(X,M)=\min\{u(X+2e_i,M),u(X+e_i,M)\}-\min\{u(X+e_i,M),u(X,M)\}-u(X+e_i,M-e_i)+u(X,M-e_i)$. We consider two cases as follows:

Case 1. $D_iu(X+e_i,M)\leq 0$, then

$A_1u(X,M)=D_iu(X+e_i,M)-D_iu(X,M-e_i)\geq D_iu(X,M)-D_iu(X,M-e_i)\geq 0$.

Case 2. $D_iu(X+e_i,M)\geq 0$, then $A_1u(X,M)\geq D_iu(X,M)-D_iu(X,M-e_i)\geq 0$.

SP2: Let $A_2(X,M)=D_iT_pu(X,M)-D_iT_pu(X,M-e_i)$. Given properties P2 and P5, we have $D_pu(X+e_i,M)\leq D_pu(X,M)\leq D_pu(X,M-e_i)$ and SP2 can be proved by considering three cases as follows:

Case 1. $D_pu(X,M-e_i)\leq 0$, then $A_2u(X,M)\geq\Delta_iu(X+e_i+e_p,M)-\Delta_iu(X+e_p,M)\geq 0$.

Case 2. $D_pu(X+e_i,M)\leq 0\leq D_pu(X,M-e_i)$, then $A_2u(X,M)\Delta_iu(X+e_i+e_p,M)-\Delta_iu(X,M)\geq\Delta_iu(X+e,M)-\Delta_iu(X,M)\geq 0$, where the third and last results are due to properties P5 and P6.

Case 3. $0\leq D_pu(X+e_i,M)$, then $A_2u(X,M)\geq=\Delta_iu(X+e_i,M)-\Delta_iu(X,M)\geq 0$.

SP3: Let$A_3u(X,M)\ =T^lu(X+e_i,M)-T^lu(X,M)\ -T^lu(X+e_i,M-e_i)+T^lu(X,M-e_i)$. Then SP3 can be proved by considering four cases as follows:

Case 1. If $\Pi_{k\neq i}x_k=0$, then $A_3u(X,M)\geq 0$.

Case 2. If $l\neq 1,x_i=0$, and $\Pi_{k\neq i}x_k\neq 0$, then we distinguish two subcases:

(1) $D_eu(X+e_i-e,M)\leq -c_l$, then we have $A_3u(X,M)\Delta_iu(X+e_i,M)-\Delta_iu(X,M)\geq 0$.
(2) $D_eu(X+e_i-e,M)\geq -c_l$, then we have$A_3u(X,M)\geq\Delta_iu(X+e_i-e,M)-\Delta_iu(X,M)\geq\Delta_iu(X,M)-\Delta_iu(X,M)=0$, which follows from property P5.

Case 3. If $l\neq 1$, and $\Pi_kx_k\neq 0$, then we have $D_eu(X-e,M-e_i)\leq D_eu(X-e,M)\ \leq D_eu(X-e+e_i,M)$, which leads to the following three subcases:

(1) $D_eu(X-e+e_i,M)\leq -c_l$, then $A_3u(X,M)\ \geq\Delta_iu(X+e_i,M)-\Delta_iu(X,\ M)\geq\ 0$.
(2) $D_eu(X-e,M-e_i)\leq -c_l\leq D_eu(X-e+e_i,M)$, then $A_3u(X,M)\geq u(X+e_i-e,M)-\ [c_l+u(X,M)]-u(X+e_i-e,M-e_i)+[c_l+u(X,M-e_i)]=\Delta_iu(X+e_i-e,M)-\Delta_iu(X,M)\geq 0$, which follows from property P5.
(3) $-c_l\leq D_eu(X-e,M-e_i)$, then $A_3u(X,M)\geq\Delta_iu(X+e_i-e,M)\ -\Delta_iu(X-\ e\ ,M)\geq 0$.

Case 4. If $l=1$, $x_i=0,\Pi_{k\neq i}x_k\neq 0$, then $A_3u(X,M)=\Delta_iu(X+e_i-e,M)-\Delta_iu(X,M)\geq 0$, which follows from property P5.

Case 5. If $l=1$, $\Pi_kx_k\neq 0$, then $A_3u(X,M)\geq\Delta_iu(X+e_i-e,M)-\Delta_iu(X-e,M)\geq 0$, which follows from property P4.

Verification of property P5

Comparing (19) and (20), we find that if the following conditions are satisfied, then property P5 holds:

$SP4:\ T_iu(X+e_j,M)-T_iu(X,M)\leq u(X+e_j,M-e_i)-u(X,M-e_i)$,
$SP5:\ D_jT_pu(X,M)-D_jT_pu(X,M-e_i)\leq 0,\quad i,p\in F$, but $i\neq p$,
$SP6:\ D_jT^lu(X,M)-D_jT^lu(X,M-e_i)\leq 0$.

SP4: Let $B_1u(X,M)=\min\{u(X+e_j+e_i,M),u(X+e_j,M)\}-\min\{u(X+e_i,M),u(X,M)\}-D_ju(X,M-e_i)$. We distinguish two cases:

Case 1. $D_iu(X,M)\geq 0$, then we have $B_1u(X,M)\leq D_ju(X,M)-D_ju(X,M-e_i)\leq 0$, where the last equation result is due to property P5.

Case 2. $D_iu(X,M)\leq 0$, then we have $B_1u(X,M)\leq D_ju(X+e_i,M)-D_ju(X,M-e_i)\leq D_ju(X,M)-D_ju(X,M-e_i)\leq 0$, which follows from property P2 and P5.

SP5: Let $B_2u(X,M)=D_jT_pu(X,M)-D_jT_pu(X,M-e_i)$. We consider two cases as follows:

Case 1. If $p=j$, we consider three cases as follows:

(1) $0\leq D_ju(X,\ M)$, then $B_2u(X,M)\ \leq D_ju(X,M)\ -D_ju(X,M-\ e_i)\leq 0$.
(2) $D_ju(X,M)\leq 0\leq D_ju(X+e_j,M-e_i)$, then $B_2u(X,M)\leq u(X+e_j,M)-u(X+e_j,M)-u(X+e_j,M-e_i)+u(X+e_j,M-e_i)=0$.
(3) $D_ju(X+e_j,M-e_i)\leq 0$, then $B_2u(X,M)\leq D_ju(X+e_j,M)-D_ju(X+e_j,M-e_i)\leq 0$.

Case 2. If $p \neq j$, we distinguish the following four subcases:

(1) $D_pu(X,M) \leq 0, D_pu(X+e_j,M-e_i) \leq 0$, then
$$B_2u(X,M) \leq D_ju(X+e_p,M)-D_ju(X+e_p,M-e_i) \leq 0.$$

(2) $D_pu(X,M) \leq 0, D_pu(X+e_j,M-e_i) \geq 0$, then
$$B_2u(X,M) \leq D_ju(X+e_p,M)-D_ju(X,M-e_i) \leq D_ju(X,M)-D_ju(X,M-e_i) \leq 0.$$

(3) $D_pu(X,M) \geq 0, D_pu(X+e_j,M-e_i) \leq 0$, then $B_2u(X,M) \leq \Delta_iu(X+e_j+e_p,M)-\Delta_iu(X,M) \leq \Delta_iu(X+e_j,M)-\Delta_iu(X,M) \leq 0$, which is due to property P5.

(4) $D_pu(X,M) \geq 0, D_pu(X+e_j,M-e_i) \geq 0$, then
$$B_2u(X,M) \leq D_ju(X,M)-D_ju(X,M-e_i) \leq 0.$$

SP6: Let $B_3u(X,M) = D_jT^lu(X,M)-D_jT^lu(X,M-e_i)$, We consider five cases as follows:

Case 1. If $\Pi_{k \neq j}x_k = 0$, then $B_3u(X,M) \leq 0$.

Case 2. If $l \neq 1$, $x_j=0$, and $\Pi_{k \neq j}x_k \neq 0$,
then we distinguish two subcases:

(1) $D_eu(X+e_j-e,M) \leq -c_l$, then $B_3u(X,M) \leq \Delta_iu(X+e_j,M)-\Delta_iu(X,M) \leq 0$.
(2) $D_eu(X+e_j-e,M) \geq -c_l$, then w get $B_3u(X,M) \leq \Delta_iu(X+e_j-e,M)-\Delta_iu(X,M) \leq \Delta_iu(X-e,M)-\Delta_iu(X,M) \leq 0$, which follows from properties P5 and P6.

Case 3. If $l \neq 1$, and $\Pi_kx_k \neq 0$, then
we have the following four subcases:

(1) $D_eu(X-e,M) \leq -c_l, D_eu(X-e+e_j,M-e_i) \leq -c_l$, then
$$B_3u(X,M) \leq \Delta_iu(X+e_j,M)-\Delta_iu(X,M) \leq 0.$$

(2) $D_eu(X-e,M) \leq -c_l, D_eu(X-e+e_i,M-e_i) \geq -c_l$, then
$$B_3u(X,M) \leq \Delta_iu(X+e_j-e,M)-\Delta_iu(X,M) \leq \Delta_iu(X-e,M) -\Delta_iu(X,M) \leq 0.$$

(3) $D_eu(X-e,M) \geq -c_l, D_eu(X-e+e_j,M-e_i) \leq -c_l$, then
$$B_3u(X,M) \leq D_ju(X-e,M)-D_ju(X,M-e_i) \leq D_ju(X,M) -D_ju(X,M-e_i) \leq 0.$$

(4) $D_eu(X-e,M) \geq -c_l, D_eu(X-e+e_i,M-e_i) \geq -c_l$, then
$$B_3u(X,M) \leq D_ju(X-e,M)-D_ju(X-e,M-e_i) \leq 0.$$

Case 4. If $l=1, x_j=0$, and $\Pi_{k \neq j}x_k \neq 0$, then we have
$B_3u(X,M)=\Delta_iu(X+e_j-e,M)-\Delta_iu(X,M) \leq \Delta_iu(X+e_j,M)-\Delta_iu(X,M) \leq 0$,

which follows from properties P5 and P6.

Case 5. If $l=1$, $\Pi_kx_k \neq 0$, then $B_3u(X,M)=\Delta_iu(X+e_j-e,M)-\Delta_iu(X-e,M) \leq 0$,
which follows from property P5.

Verification of property P6

If we can prove the following observations, then property P6 holds.

$SP7: \ T_iu(X+e,M)-T_iu(X,M) \geq u(X+e,M-e_i)-u(X,M-e_i)$,
$SP8: \ D_eT_pu(X,M)-D_eT_pu(X,M-e_i) \geq 0, \quad i,p \in F$, but $i \neq p$,
$SP9: \ D_eT^lu(X,M)-D_eT^lu(X,M-e_i) \geq 0$.

SP7: Let $C_1u(X,M)=min\{u(X+e+e_i,M),u(X+e,M)\}-min\{u(X+e_i,M),u(X,M)\} -D_eu(X,M-e_i)$.
We consider two cases as follows:

Case 1. $D_iu(X+e,M) \leq 0$, then
$C_1u(X,M) \geq D_eu(X+e_i,M)-D_eu(X,M-e_i) \geq D_eu(X,M)-D_eu(X,M-e_i) \geq 0$.
Case 2. $D_iu(X+e_i,M) \geq 0$, then $C_1u(X,M) \geq D_eu(X,M)-D_iu(X,M-e_i) \geq 0$.

SP8: Let $C_2(X,M)=D_eT_pu(X,M)-D_eT_pu(X,M-e_i)$. We consider four cases as follows:

Case 1. $D_pu(X+e,M) \leq 0, D_pu(X,M-e_i) \leq 0$, then
$C_2u(X,M) \geq D_eu(X+e_p,M)-D_eu(X+e_p,M-e_i) \geq 0$.
Case 2. $D_pu(X+e,M) \leq 0, D_pu(X,M-e_i) \geq 0$, then
$C_2u(X,M) \geq D_eu(X+e_p,M)-D_eu(X,M-e_i) \geq D_eu(X,M)-D_eu(X,M-e_i) \geq 0$.
Case 3. $D_pu(X+e,M) \geq 0, D_pu(X,M-e_i) \leq 0$, then $C_2u(X,M) \geq \Delta_iu(X+e,M)-\Delta_iu(X+e_p,M) \geq \Delta_iu(X,M)-\Delta_iu(X+e_p,M) \geq 0$,
which follows from properties P5 and P6.

Case 4. $D_pu(X+e,M) \geq 0, D_pu(X,M-e_i) \geq 0$, then
$C_2u(X,M) \geq D_eu(X,M)-D_eu(X,M-e_i) \geq 0$.

SP9: Let $C_3u(X,M) = T^lu(X+e,M) - T^lu(X,M) - T^lu(X+e,M-e_i) + T^lu(X,M-e_i)$, then SP9 can be proved by considering the following four cases:

Case 1. If $l \neq 1$, and $\Pi_kx_k=0$, then there are two subcases:

(1) $D_eu(X,M) \leq -c_l$, the we have $C_3u(X,M) \geq c_l+u(X+e,M)-[c_l+u(X+e,M-e_i)]-\Delta_iu(X,M)=\Delta_iu(X+e,M)-\Delta_iu(X,M) \geq 0$.
(2) $D_eu(X,M) \geq -c_l$, then $C_3u(X,M) \geq u(X,M)-u(X,M-e_i)-\Delta_iu(X,M) = 0$.

Case 2. If $l \neq 1$, and $\Pi_kx_k \neq 0$, then it leads to three subcases by considering $D_eu(X-e,M-e_i) \leq D_eu(X-e,M) \leq D_eu(X,M)$.

(1) $D_eu(X-e,M-e_i) \leq D_eu(X,M) \leq -c_l$, then
$$C_3u(X,M) \geq [c_l+u(X+e,M)]-[c_l+u(X,M)]-[c_l+u(X+e,M-e_i)] +[c_l+u(X,M-e_i)] = \Delta_iu(X+e,M)-\Delta_iu(X,M) \geq 0.$$

(2) $D_eu(X-e,M-e_i) \leq -c_l \leq D_eu(X,M)$, then
$$C_3u(X,M) \geq u(X,M)-[c_l+u(X,M)]-u(X,M-e_i)]+[c_l+u(X,M-e_i)] = 0.$$

(3) $-c_l \leq D_eu(X-e,M-e_i) \leq D_eu(X,M)$, then
$$C_3u(X,M) \geq \Delta_iu(X,M)-\Delta_iu(X-e,M) \geq 0.$$

Case 3. If $l \neq 1$, and $\Pi_kx_k = 0$, then we have
$C_3u(X,M) = u(X,M)-u(X-e,M)-u(X,M-e_i)-\Delta_iu(X,M) = 0$.
Case 4. If $l=1$,and $\Pi_kx_k \neq 0$, then we have
$C_3u(X,M) = \Delta_iu(X,M)-\Delta_iu(X-e,M) \geq 0$. □

Lemma 1 shows that operator T defined following the same logic of the optimal equation preserves properties P1–P7, so we have $J^{\pi^*}(X,M) \in V$.

Property P1 indicates that $D_iJ^{\pi^*}(X,M), i = 1,2,\ldots,m$, is increasing in X_i such that for any $X_i \geq S_i^*(X_{-i},M_{-i})$, $D_iJ^{\pi^*}(X,M) \geq 0$, so not to produce is optimal. Hence the optimal production policy is a base-stock policy. P3 indicates that $D_eJ^{\pi^*}(X,M)$ is increasing in X_i, $i=1,2,\ldots,m$, such that for any $X_i \geq R_{i,l}^*(X_{-i},M_{-i})$, $D_eJ^{\pi^*}(X,M) \geq -c_l$, so it is optimal to satisfy a demand from class l. Hence the rationing policy with state-dependent rationing levels is optimal.

Inequalities (6) and (7) can be induced directly from properties P2 and P5. If it is optimal to produce component i when the system is in state (X,M), i.e., $D_iJ^{\pi^*}(X,M) \leq 0$, then from property P2, we have $D_iJ^{\pi^*}(X+e_j,M) \leq D_iJ^{\pi^*}(X,M) \leq 0, i \neq j$, which leads to (6). From property P5, we have $D_iJ^{\pi^*}(X,M+e_j) \leq D_iJ^{\pi^*}(X,M) \leq 0, \quad m_j = 0, i \neq j$, which leads to (7). Inequalities (8) and (9) can be induced directly from properties P3 and P6. If it is optimal to satisfy a demand from class l in state (X,M), i.e., $D_eJ^{\pi^*}(X,M) \geq -c_l$, then from property P3, we have $D_eJ^{\pi^*}(X+e_j,M) \geq D_eJ^{\pi^*}(X,M) \geq -c_l$, $i \neq j$, which leads to (8). From property P6, we have $D_eJ^{\pi^*}(X,M+e_j) \geq D_eJ^{\pi^*}(X,M) \geq -c_l$, $m_j = 0$, which leads to (9). Inequality (10) is obtained directly from properties P1 and P7.

Proposition 1 is proved. □

Proof of Proposition 2

First we prove the existence of the average cost. If the following two conditions are satisfied, then there exists an optimal constant average cost g, which is independent of the starting state (see, e.g., Weber and Stidham 1987, Puterman, 1994; Benjaafar and El Hafsi, 2006): (1) there exists a stationary policy π', which induces a positive recurrent Markov chain and has a finite average cost $g^{\pi'}$ and (2) the set $X \in Z^+ : h(X) < g^{\pi'}$ is not empty and finite.

Consider a stationary policy π', which operates in the following manner: The inventory of component $i, i=1,2, \ldots, m$, is controlled by a base-stock policy with the base-stock level s_i and demands are satisfied on the FCFS basis. It is easy to check that the induced process under policy π' is a positive recurrent Markov chain, so the system has a finite constant average cost $g^{\pi'}$. As for condition (2), $h(X)$ is an increasing convex function in the component inventory level $X_i, i=1,2,\ldots,m$, so the number of X that satisfies $h(X) < g^{\pi'}$ is non-empty and finite. Based on the above analysis, there exists a vector $f(X,M)$ satisfying the optimality equation under the average cost criterion, i.e.,

$$f(X,M) = \frac{1}{v}\left[Tf(X,M) - g\right], \tag{19}$$

where $v = \sum_{i=1}^{m}(\mu_i + b_i + r_i) + \sum_{j=1}^{n}\lambda_j$. The structural properties of the optimal control policy are determined through the vector $f(X,M)$. To simplify notation, we define a new operator T' on the set of real-valued function $u(X,M)$ defined on the state space Ω, where $T'u(X,M) = (1/v)[Tu(X,M) - g]$.

With respect to the average cost criterion, we do not re-scale the time unit in the optimality equation (14) for the average cost case. Lemma 1 states that if $f(X,M) \in V$, then operator T possesses properties P1 to P6. A linear transformation of operator T, T' preserves these properties. While the showing of property P7 in Lemma 1 used the relationship $\beta + \sum_{i=1}^{m}(\mu_i + b_i + r_i) + \sum_{j=1}^{n}\lambda_j = 1$, we can prove property P7 in a similar way as follows:

$$D_eT'f(X-e,M) = \frac{1}{v}\left[Tf(X,M) - Tf(X,M)\right] \geq \frac{1}{v}\left[h(X) - h(X-e) - vc_1\right] \geq -c_1,$$

which follows from the assumption that $h(X)$ is a positive and increasing convex function and the definition of operator T given in (6). Therefore if we have $f(X,M) \in V$, then $T'f(X,M) \in V$, which indicates that operator T' possesses properties P1–P7. That is, the optimal control policy retains the same structural properties as those under the expected total discounted cost criterion. □

Proof of Proposition 3

In order to prove Proposition 3, we verify that the following Lemma 2 holds. To facilitate analysis, we define a real-value function set $\ddot{u}$ on the state space Θ. If a function $\gamma(Y,M) \in \ddot{u}$, then $\gamma(Y,M)$ satisfies properties P1–P6.

Lemma 2. If $\gamma(Y,M) \in \ddot{u}$, then $\tilde{T}\gamma(Y,M) \in \Upsilon$.

Proof. The proofs of properties P1–P3 are similar to those in Benjaafar and El Hafsi (2006) and the proofs of properties P4–P6 are similar to those of Proposition 1. We omit the proofs from the paper for the sake of brevity. □

Then we have $G(Y,M) \in \ddot{u}$ and Proposition 3 is obtained directly from Lemma 2. □

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按单装配系统中组件生产和库存分配控制策略研究

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摘 要 针对由两种组件、三类顾客需求组成的按单装配系统, 本文研究了其中的组件生产控制与库存分配问题. 在各类顾客需求是泊松到达过程, 各种组件加工时间服从指数分布的假设下, 我们运用马尔科夫决策理论建立了无限期折扣总成本模型, 根据 Lippman 转换得到了相应归一化后的离散最优方程, 在此基础之上分析了生产和库存分配联合最优控制策略的结构性质. 本文证明了最优策略是依赖于系统状态的动态策略. 组件的最优生产策略是动态基库存策略, 其中基库存水平是关于系统中其他组件库存水平的非减函数. 而最优的分配策略是动态的阈值策略, 对于只需一种组件构成的顾客需求, 组件的分配阈值是系统中另一组件库存水平的增函数; 而对于同时需要两种组件组成的顾客需求, 其各组件的分配阈值是另一组件库存水平的减函数. 最后通过数值试验给出了各个参数对联合最优控制策略的影响, 并得到了相应的管理启示.

关键词 按单装配, 多类需求, 马尔科夫决策, 最优控制策略

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Joint Control of Component Production and Inventory Allocation in an Assemble-to-order System with Lost Sales

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Abstract This paper considers a joint control problem of combined component production and inventory allocation in an assemble-to-order system which consists of two components and three demand classes with lost sales. Each demand class arrives according to a Poisson process, and the production time of each component follows an exponential distribution. By formulating the system as a Markov decision process under the expected total discounted cost criterion, we obtain the optimality equation following the Lippman transformation, from which we derive the structural properties of the optimal control policy. Specially, the optimal production policy for each component is shown to be a base stock policy with the base-stock level non-decreasing in the inventory level of the other component, and the optimal inventory allocation for each component is a state-dependent threshold policy, where the threshold point for the demand for one kind of components is non-decreasing in the inventory level of the other component, while the threshold point for the demand for both components is non-increasing in the inventory level of the other component. Finally, we give some numerical examples to show how the optimal control policy changes with the system parameters, and we also provide some managerial insights.

Key words Assemble-to-order system, multi-class demand, Markov decision process, optimal control policy

在产品更新换代频繁的今天, 如何以更低的成本满足市场日益多样化需求成为企业管理者面临的难题之一, 也是学术界关注的主要问题之一. 在这种背景下, 按单装配 (Assemble-to-order, ATO) 这种新的运营策略被提出, 并且被许多企业如著名的代工企业 Flextronics 公司采用[1]. 采用 ATO 运营方式的企业, 按照存货生产方式提前生产通用组件, 而产成品在接到顾客订单以后再进行个性化装配 (按单装配系统的结构图如图 1 所示). ATO 运营方式使得企业能够以较低的库存量、较快的速度提供多样化的产品满足顾客的个性化需求. 然而除组件的生产控制这个传统问题外, 通用件的库存分配问题也十分重要. 采用有效的生产策略和库存分配策略可以更好地体现 ATO 运营方式的优越性, 并且与之相关的问题也引起了学术界的研究兴趣. 作为一个制造大国, 以华为等为代表的我国企业也在应用这种策略来参与国际市场竞争.

目前与本文所研究 ATO 运营问题相关的文献大致可以分为两类: 一类是外生提前期情况下的库存控制问题[2−6]. 本文主要关注点是内生提前期方面的工作. Ha[7] 研究了单产品, 多类需求、缺货不补 (Lost sales) 的库存生产 (Make-to-stock, MTS) 系统, 在各类需求以泊松过程到达, 产品加工时间服从指数分布假设下, 证明了最优生产策略是静态基库存策略、最优库存分配策略是阈值策略. Ha[8] 研究了两类需求、缺货候补系统 (Backorders), 证明了最优生产策略是基库存策略、最优库存分配策略是依赖于系统中各类需求缺货量的动态策略. 2000 年 Ha[9] 将原模型进一步推广到产品加工时间服从埃尔兰分布的情况, 证明了最优生产策略是依赖于系

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统当前库存水平和生产进度的基库存策略; 而最优库存分配策略是依赖于系统当前库存水平和生产进度的阈值策略. 2009 年 Gayon 等[10] 在 Ha[7] 模型的基础上考虑了各类顾客向供应商提供一个不完美的需求信息的情形. 并给出了最优生产策略和库存分配策略. 2002 年 Vericourt 等[11] 将 Ha[8] 推广到多类需求情况, 并证明了最优策略具有和 Ha[8] 类似的性质. Benjaafar 等[1] 研究了多组件、单产品、多类需求的 ATO 系统. 在各类需求以相互独立泊松过程到达、组件加工时间服从指数分布假设下, 他们证明了组件最优生产策略是动态基库存策略、组件库存最优分配策略则是动态阈值策略.

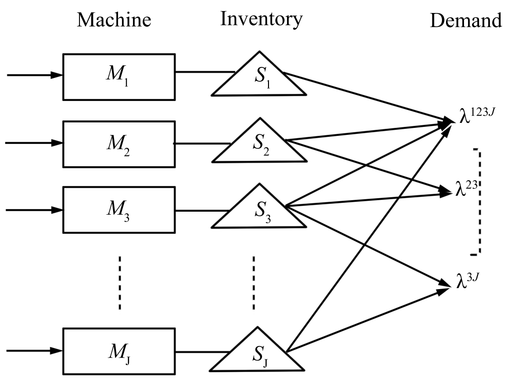

图 1　ATO 系统结构图

Fig. 1　The structure of ATO system

上述研究的是单产品、多类顾客的 MTS 系统和 ATO 系统, 本文研究的系统与其不同, 研究的是产品需求缺货不补的两种组件、三类产品需求的 ATO 系统. 我们将运用马尔科夫决策理论、数值试验等方法研究系统最优策略的结构性质.

1　模型建立与分析

本文研究这样一种 ATO 系统: 两种组件分别在两个不同的设备上生产并且分别存入库存点 1、2 两处, 在接到三类不同顾客订单以后, 分别用这两种组件装配成三种产品来满足这三类顾客需求. 第 1 类产品需求由 1 个组件 1 构成, 第 2 类产品需求由 1 个组件 2 构成, 第 3 类产品需求由 1 个组件 1 和 1 个组件 2 构成. 假定各类需求以相互独立的泊松过程到达, 到达率分别为 $\lambda_1, \lambda_2, \lambda_3$, 组件生产时间分别服从均值为 $1/\mu_1, 1/\mu_2$ 的指数分布. 系统运营的主要问题是组件的生产决策和组件的库存分配决策. 生产决策确定各组件何时生产、生产到什么时刻; 库存分配决策确定两种组件库存如何在三类产品需求中分配, 即决定是否满足某个到达的产品需求. 如果需求被拒绝, 将会产生拒绝成本, 三类需求的单位拒绝成本记为 c_1, c_2, c_3. 不失一般性, 我们假定 $c_3 > c_1 \geq c_2$.

以 $\boldsymbol{x}(t) = (x_1(t), x_2(t))$ 表示系统在时刻 t 的状态, 其中 $x_i(t)$ 表示时刻 t 第 i 种组件的库存水平. $h(\boldsymbol{x}(t)) = h_1(x_1) + h_2(x_2)$ 表示系统在时刻 t 状态 $\boldsymbol{x}(t)$ 下的库存持有成本, $h_i(x_i)$ 是关于 x_i 单调增的凸函数. 由于各类需求到达的时间间隔和单位组件生产所需时间都服从指数分布, 所以问题可以看作一个马尔科夫决策过程.

设 π 是任意可行控制策略, 系统所处状态记为 (x_1, x_2), 即 $x_1(t) = x_1, x_2(t) = x_2$. 在策略 π 下, 决策者行动记为 $a^\pi(x_1, x_2) = (u_1, u_2, v_1, v_2, v_3)$, 如果 $u_i = 1$, 那么当系统处于状态 (x_1, x_2) 时, 需要生产第 i 类组件, $u_i = 0$ 则不生产; 如果 $v_i = 1$, 那么当系统处于状态 (x_1, x_2) 时, 满足到达的第 i 类需求, $v_i = 0$ 则拒绝该需求. $N_i^\pi(t)$ 表示在控制策略 π 下, $[0, t]$ 时间段内被拒绝的第 i 类需求的总数. 系统的期望无限期折扣总成本可以写为

$$\mathrm{E}_{\boldsymbol{x}}^{\pi}\left[\int_0^\infty \mathrm{e}^{-\alpha t} h(\boldsymbol{x}(t))\mathrm{d}t + \sum_{i=1}^{3}\int_0^\infty \mathrm{e}^{-\alpha t} c_i \mathrm{d}N_i^\pi(t)\right] \tag{1}$$

其中 α 表示折扣因子, $\boldsymbol{x} = \boldsymbol{x}(0)$ 表示初始库存水平. 式 (1) 中, 第一部分表示的是无限期折扣的总持有成本, 第二部分表示的是无限期折扣的总拒绝成本. 它们都与所采取的策略 π 有关. 我们的目标是寻找一个最优控制策略 π^*, 使得上述期望总成本最小, 记 $V(\boldsymbol{x})$ 为上述最小成本. 给定控制策略 π, 库存水平的变化过程是连续时间、离散状态的马尔科夫过程. 根据文献 [12], 对系统状态转移概率进行归一化处理, 设 $\gamma = \sum_{i=1}^{3}\lambda_i + \sum_{i=1}^{2}\mu_i$, 重新定义时间单位, 使得 $\alpha + \gamma = 1$, 对式 (1) 进行离散化处理后可得如下离散化的最优动态方程:

$$V(\boldsymbol{x}) = h(\boldsymbol{x}) + \sum_{k=1}^{2}\mu_k T_k V(\boldsymbol{x}) + \sum_{i=1}^{3}\lambda_i T^i V(\boldsymbol{x}) \tag{2}$$

其中

$$T_k V(\boldsymbol{x}) = \min\{V(\boldsymbol{x} + \boldsymbol{e}_k), V(\boldsymbol{x})\}$$

$$T^i V(\boldsymbol{x}) = \begin{cases} V(\boldsymbol{x}) + c_i, & x_i = 0 \\ \min\{V(\boldsymbol{x} - \boldsymbol{e}_i), V(\boldsymbol{x}) + c_i\}, & x_i \neq 0 \end{cases}$$

$$\boldsymbol{e}_1 = (1, 0),\ \boldsymbol{e}_2 = (0, 1),\ \boldsymbol{e}_3 = (1, 1)$$

最优方程 (2) 表示: 以当前状态作为初始状态的无限期折扣总成本 (等式左边) 等于当前时刻到下一次状态转移时刻期间所产生的持有成本的现值, 加上以转移后状态作为初始状态的无限期折扣成本的现值. 最优策略满足最优方程 (2), 而从最优方程可以刻画最优策略如下:

1) 如果 $V(\boldsymbol{x} + \boldsymbol{e}_i) - V(\boldsymbol{x}) \leq 0$, 即某一组件库存水平增加 1 个单位可以降低系统的最小成本, 则应该选择开始生产该组件; 否则停止生产.

2) 当有现货时, 如果 $V(\boldsymbol{x})+c_i \ge V(\boldsymbol{x}-\boldsymbol{e}_i)$, 即某组件库存水平减少 1 个单位使得系统最小成本增加的量小于拒绝只需要该种组件的成品需求所将产生的拒绝成本时, 应满足该类需求; 否则拒绝. 如果组件 1 和 2 各减少一个单位使得系统最小成本增加的总量小于拒绝第三类成品需求所将产生的拒绝成本, 则应满足该类需求; 否则拒绝.

2 最优控制策略及其结构性质

记 $\boldsymbol{e}=\boldsymbol{e}_3$, 定义如下微分算子:

$$\Delta_j V(\boldsymbol{x}) = V(\boldsymbol{x}+\boldsymbol{e}_j) - V(\boldsymbol{x}) \tag{3}$$

$$\Delta_{\boldsymbol{e}} V(\boldsymbol{x}) = V(\boldsymbol{x}+\boldsymbol{e}) - V(\boldsymbol{x}) \tag{4}$$

$$\Delta_{i,j} V(\boldsymbol{x}) = \Delta_j V(\boldsymbol{x}+\boldsymbol{e}_i) - \Delta_j V(\boldsymbol{x}) \tag{5}$$

以 $\mathcal{V}$ 表示定义在非负整数集上的函数集合, 并且满足以下性质: 若 $V \in \mathcal{V}$, 则对于 $i=1,2$ 有如下性质:

$$\text{A1}: \quad \Delta_{i,i} V(\boldsymbol{x}) \ge 0 \tag{6}$$

$$\text{A2}: \quad \Delta_{1,2} V(\boldsymbol{x}) \le 0 \tag{7}$$

$$\text{A3}: \quad \Delta_{\boldsymbol{e},i} V(\boldsymbol{x}) \ge 0 \tag{8}$$

性质 A1 表明 $V(\boldsymbol{x})$ 具有凸性, 性质 A2 表明 $\Delta_i V(\boldsymbol{x})$ 是 x_j 的减函数, 性质 A3 表明 $\Delta_e V(\boldsymbol{x})$ 是 x_i 的增函数.

引理 1. 如果 $V \in \mathcal{V}$, 则有 $TV \in \mathcal{V}$, 其中 $TV(\boldsymbol{x}) = h(\boldsymbol{x}) + \sum_{k=1}^{2} \mu_k T_k V(\boldsymbol{x}) + \sum_{i=1}^{3} \lambda_i T^i V(\boldsymbol{x})$.

证明. 由文献 [1] 知当 $V \in \mathcal{V}$ 时有 $T_k V(\boldsymbol{x}) \in \mathcal{V}$ 以及 $T^3 V(\boldsymbol{x}) \in \mathcal{V}$. 由对称性, 只需证明 $T^1 V(\boldsymbol{x}) \in \mathcal{V}$, 即证明 $T^1 V(\boldsymbol{x})$ 满足性质 A1 ∼ A3.

$$T^1V(\boldsymbol{x}) = \begin{cases} V(\boldsymbol{x})+c_1, & x_1=0 \\ \min\{V(\boldsymbol{x}-\boldsymbol{e}_1), V(\boldsymbol{x})+c_1\}, & x_1 \ne 0 \end{cases} = V(\boldsymbol{x})+c_1+ \begin{cases} 0, & x_1=0 \\ \min\{-\Delta_1 V(\boldsymbol{x}-\boldsymbol{e}_1)-c_1, 0\}, & x_1 \ne 0 \end{cases}$$

1) 要证明 T^1V 满足性质 A1, 即证 $\Delta_{i,i} T^1 V(\boldsymbol{x}) \ge 0$, 当 $i=1$ 时, 由文献 [7] 知结论成立. 下面证明 $\Delta_{2,2} T^1 V(\boldsymbol{x}) \ge 0$.

$$\Delta_{2,2}T^1V(\boldsymbol{x}) = \Delta_{2,2}V(\boldsymbol{x}) + \begin{cases} 0, & x_1=0 \\ \min\{-\Delta_1V(\boldsymbol{x}-\boldsymbol{e}_1+2\boldsymbol{e}_2)-c_1,0\} - \\ \quad 2\min\{-\Delta_1V(\boldsymbol{x}-\boldsymbol{e}_1+\boldsymbol{e}_2)-c_1,0\} + \\ \quad \min\{-\Delta_1V(\boldsymbol{x}-\boldsymbol{e}_1)-c_1,0\}, & x_1 \ne 0 \end{cases}$$

当 $x_1=0$ 时, $\Delta_{2,2}T^1V(\boldsymbol{x}) = \Delta_{2,2}V(\boldsymbol{x}) \ge 0$.

当 $x_1 \ne 0$ 时, 由 A2 知:

$$-\Delta_1V(\boldsymbol{x}-\boldsymbol{e}_1+2\boldsymbol{e}_2)-c_1 \ge -\Delta_1V(\boldsymbol{x}-\boldsymbol{e}_1+\boldsymbol{e}_2)-c_1 \ge -\Delta_1V(\boldsymbol{x}-\boldsymbol{e}_1)-c_1$$

下面分四种情况分别进行分析:

a) 当 $-\Delta_1V(\boldsymbol{x}-\boldsymbol{e}_1)-c_1 \ge 0$ 时,

$$\Delta_{2,2}T^1V(\boldsymbol{x}) = \Delta_{2,2}V(\boldsymbol{x}) \ge 0$$

b) 当 $-\Delta_1V(\boldsymbol{x}-\boldsymbol{e}_1+\boldsymbol{e}_2)-c_1 \ge 0 \ge -\Delta_1V(\boldsymbol{x}-\boldsymbol{e}_1)-c_1$ 时,

$$\Delta_{2,2}T^1V(\boldsymbol{x}) = \Delta_{2,2}V(\boldsymbol{x}) - \Delta_1V(\boldsymbol{x}-\boldsymbol{e}_1)-c_1 = \Delta_{2,\boldsymbol{e}}V(\boldsymbol{x}-\boldsymbol{e}_1) - \Delta_1V(\boldsymbol{x}-\boldsymbol{e}_1+\boldsymbol{e}_2)-c_1 \ge 0$$

c) 当 $-\Delta_1V(\boldsymbol{x}-\boldsymbol{e}_1+2\boldsymbol{e}_2)-c_1 \ge 0 \ge -\Delta_1V(\boldsymbol{x}-\boldsymbol{e}_1+\boldsymbol{e}_2)-c_1$ 时,

$$\Delta_{2,2}T^1V(\boldsymbol{x}) = \Delta_{2,\boldsymbol{e}}V(\boldsymbol{x}-\boldsymbol{e}_1) + \Delta_1V(\boldsymbol{x}-\boldsymbol{e}_1+\boldsymbol{e}_2)+c_1 \ge 0$$

d) 当 $0 \ge -\Delta_1V(\boldsymbol{x}-\boldsymbol{e}_1+2\boldsymbol{e}_2)-c_1$ 时,

$$\Delta_{2,2}T^1V(\boldsymbol{x}) = \Delta_{2,2}V(\boldsymbol{x}) - \Delta_1V(\boldsymbol{x}-\boldsymbol{e}_1+2\boldsymbol{e}_2) + 2\Delta_1V(\boldsymbol{x}-\boldsymbol{e}_1+\boldsymbol{e}_2) - \Delta_1V(\boldsymbol{x}-\boldsymbol{e}_1) = \Delta_{2,2}V(\boldsymbol{x}-\boldsymbol{e}_1) \ge 0$$

2) 要证明 T^1V 满足性质 A2, 即证 $\Delta_{1,2}T^1V(\boldsymbol{x}) \le 0$.

$$\Delta_{1,2}T^1V(\boldsymbol{x}) = \Delta_{1,2}V(\boldsymbol{x}) + \begin{cases} \min\{-\Delta_1V(\boldsymbol{x}+\boldsymbol{e}_2)-c_1,0\} - \\ \quad \min\{-\Delta_1V(\boldsymbol{x})-c_1,0\}, & x_1=0 \\ \min\{-\Delta_1V(\boldsymbol{x}+\boldsymbol{e}_2-\boldsymbol{e}_1)-c_1,0\} - \\ \quad \min\{-\Delta_1V(\boldsymbol{x})-c_1,0\} - \\ \quad \min\{-\Delta_1V(\boldsymbol{x}-\boldsymbol{e}_1)-c_1,0\} + \\ \quad \min\{-\Delta_1V(\boldsymbol{x}+\boldsymbol{e}_2)-c_1,0\}, & x_1 \ne 0 \end{cases}$$

由 A2 和 A3 知: $-\Delta_1V(\boldsymbol{x}+\boldsymbol{e}_2-\boldsymbol{e}_1)-c_1 \ge -\Delta_1V(\boldsymbol{x}-\boldsymbol{e}_1)-c_1 \ge -\Delta_1V(\boldsymbol{x}+\boldsymbol{e}_2)-c_1 \ge -\Delta_1V(\boldsymbol{x})-c_1$.

下面分五种情况分别进行分析:

a) 当 $-\Delta_1V(\boldsymbol{x})-c_1 \ge 0$ 时,

$$\Delta_{1,2}T^1V(\boldsymbol{x}) = \Delta_{1,2}V(\boldsymbol{x}) \le 0$$

b) 当 $-\Delta_1V(\boldsymbol{x}+\boldsymbol{e}_2)-c_1 \ge 0 \ge -\Delta_1V(\boldsymbol{x})-c_1$ 时,

$$\Delta_{1,2}T^1V(\boldsymbol{x}) = \Delta_{1,2}V(\boldsymbol{x}) + \Delta_1V(\boldsymbol{x})+c_1 = \Delta_1V(\boldsymbol{x}+\boldsymbol{e}_2)+c_1 \le 0$$

c) 当 $-\Delta_1 V(\boldsymbol{x}-\boldsymbol{e}_1)-c_1 \geq 0 \geq -\Delta_1 V(\boldsymbol{x}+\boldsymbol{e}_2)-c_1$ 时,

$$\Delta_{1,2} T^1 V(\boldsymbol{x}) = \Delta_{1,2} V(\boldsymbol{x}) - \Delta_1 V(\boldsymbol{x}+\boldsymbol{e}_2) + \Delta_1 V(\boldsymbol{x}) \leq 0$$

d) 当 $-\Delta_1 V(\boldsymbol{x}+\boldsymbol{e}_2-\boldsymbol{e}_1)-c_1 \geq 0 \geq -\Delta_1 V(\boldsymbol{x}-\boldsymbol{e}_1)-c_1$ 时,

$$\Delta_{1,2} T^1 V(\boldsymbol{x}) = -\Delta_1 V(\boldsymbol{x}-\boldsymbol{e}_1) - c_1 \leq 0$$

e) 当 $0 \geq -\Delta_1 V(\boldsymbol{x}+\boldsymbol{e}_2-\boldsymbol{e}_1)-c_1$ 时,

$$\begin{aligned}\Delta_{1,2} T^1 V(\boldsymbol{x}) = {} & \Delta_{1,2} V(\boldsymbol{x}) - \\ & \Delta_1 V(\boldsymbol{x}+\boldsymbol{e}_2) + \Delta_1 V(\boldsymbol{x}) - \Delta_1 V(\boldsymbol{x}-\boldsymbol{e}_1) - \\ & \Delta_1 V(\boldsymbol{x}-\boldsymbol{e}_1+\boldsymbol{e}_2) = \Delta_{1,2} V(\boldsymbol{x}-\boldsymbol{e}_1) \leq 0\end{aligned}$$

可知 $x_1=0$ 及 $x_1 \neq 0$ 时都有 $\Delta_{1,2} T^1 V(\boldsymbol{x}) \leq 0$.

3) 要证明 $T^1 V$ 满足性质 A3, 即证 $\Delta_{i,\boldsymbol{e}} T^1 V(\boldsymbol{x}) \geq 0$, $i=1,2$.

$$\Delta_{1,\boldsymbol{e}} T^1 V(\boldsymbol{x}) = \Delta_{1,\boldsymbol{e}} V(\boldsymbol{x}) + \begin{cases} \min\{-\Delta_1 V(\boldsymbol{x}+\boldsymbol{e})-c_1, 0\} - \\ \quad \min\{-\Delta_1 V(\boldsymbol{x}+\boldsymbol{e}_2)-c_1, 0\} - \\ \quad \min\{-\Delta_1 V(\boldsymbol{x})-c_1, 0\}, & x_1 = 0 \\ \min\{-\Delta_1 V(\boldsymbol{x}+\boldsymbol{e})-c_1, 0\} - \\ \quad \min\{-\Delta_1 V(\boldsymbol{x}+\boldsymbol{e}_2)-c_1, 0\} - \\ \quad \min\{-\Delta_1 V(\boldsymbol{x})-c_1, 0\} + \\ \quad \min\{-\Delta_1 V(\boldsymbol{x}-\boldsymbol{e}_1)-c_1, 0\}, & x_1 \neq 0 \end{cases}$$

由 A2 和 A3 知: $-\Delta_1 V(\boldsymbol{x}-\boldsymbol{e}_1)-c_1 \geq -\Delta_1 V(\boldsymbol{x}+\boldsymbol{e}_2)-c_1 \geq -\Delta_1 V(\boldsymbol{x})-c_1 \geq -\Delta_1 V(\boldsymbol{x}+\boldsymbol{e})-c_1$.

下面分四种情况分别进行分析:

a) 当 $-\Delta_1 V(\boldsymbol{x}+\boldsymbol{e})-c_1 \geq 0$ 时,

$$\Delta_{1,\boldsymbol{e}} T^1 V(\boldsymbol{x}) = \Delta_{1,\boldsymbol{e}} V(\boldsymbol{x}) \geq 0$$

b) 当 $-\Delta_1 V(\boldsymbol{x})-c_1 \geq 0 \geq -\Delta_1 V(\boldsymbol{x}+\boldsymbol{e})-c_1$ 时,

$$\begin{aligned}\Delta_{1,\boldsymbol{e}} T^1 V(\boldsymbol{x}) = {} & \Delta_{1,\boldsymbol{e}} V(\boldsymbol{x}) - \Delta_1 V(\boldsymbol{x}+\boldsymbol{e}) - c_1 = \\ & \Delta_1 V(\boldsymbol{x}) - c_1 \geq 0\end{aligned}$$

c) 当 $-\Delta_1 V(\boldsymbol{x}+\boldsymbol{e}_2)-c_1 \geq 0 \geq -\Delta_1 V(\boldsymbol{x})-c_1$ 时,

$$\Delta_{1,\boldsymbol{e}} T^1 V(\boldsymbol{x}) = \Delta_{1,\boldsymbol{e}} V(\boldsymbol{x}) - \Delta_1 V(\boldsymbol{x}+\boldsymbol{e}) + \Delta_1 V(\boldsymbol{x}) = 0$$

d) 当 $-\Delta_1 V(\boldsymbol{x}-\boldsymbol{e}_1)-c_1 \geq 0 \geq -\Delta_1 V(\boldsymbol{x}+\boldsymbol{e}_2)-c_1$ 时,

$$\Delta_{1,\boldsymbol{e}} T^1 V(\boldsymbol{x}) = \Delta_1 V(\boldsymbol{x}+\boldsymbol{e}_2) + c_1 \geq 0$$

e) 当 $0 \geq -\Delta_1 V(\boldsymbol{x}-\boldsymbol{e}_1)-c_1$ 时,

$$\Delta_{1,\boldsymbol{e}} T^1 V(\boldsymbol{x}) = \Delta_{1,\boldsymbol{e}} V(\boldsymbol{x}-\boldsymbol{e}_1) \geq 0$$

可知 $x_1=0$ 及 $x_1 \neq 0$ 时都有 $\Delta_{1,\boldsymbol{e}} T^1 V(\boldsymbol{x}) \geq 0$.

$$\Delta_{2,\boldsymbol{e}} T^1 V(\boldsymbol{x}) = \Delta_{2,\boldsymbol{e}} V(\boldsymbol{x}) + \begin{cases} \min\{-\Delta_1 V(\boldsymbol{x}+2\boldsymbol{e}_2)-c_1, 0\} - \\ \quad \min\{-\Delta_1 V(\boldsymbol{x}+\boldsymbol{e}_2)-c_1, 0\}, & x_1 = 0 \\ \min\{-\Delta_1 V(\boldsymbol{x}+2\boldsymbol{e}_2)-c_1, 0\} - \\ \quad \min\{-\Delta_1 V(\boldsymbol{x}+\boldsymbol{e}_2)-c_1, 0\} - \\ \quad \min\{-\Delta_1 V(\boldsymbol{x}+\boldsymbol{e}_2-\boldsymbol{e}_1)-c_1, 0\} + \\ \quad \min\{-\Delta_1 V(\boldsymbol{x}-\boldsymbol{e}_1)-c_1, 0\}, & x_1 \neq 0 \end{cases}$$

由 A2 和 A3 知: $-\Delta_1 V(\boldsymbol{x}+\boldsymbol{e}_2-\boldsymbol{e}_1)-c_1 \geq \begin{cases} -\Delta_1 V(\boldsymbol{x}-\boldsymbol{e}_1)-c_1 \\ -\Delta_1 V(\boldsymbol{x}+2\boldsymbol{e}_2)-c_1 \end{cases} \geq -\Delta_1 V(\boldsymbol{x}+\boldsymbol{e}_2)-c_1$,

下面分六种情况分别进行分析:

a) 当 $-\Delta_1 V(\boldsymbol{x}+\boldsymbol{e}_2)-c_1 \geq 0$ 时,
$\Delta_{2,\boldsymbol{e}} T^1 V(\boldsymbol{x}) = \Delta_{2,\boldsymbol{e}} V(\boldsymbol{x}) \geq 0$

b) 当 $\begin{cases} -\Delta_1 V(\boldsymbol{x}-\boldsymbol{e}_1)-c_1 \\ -\Delta_1 V(\boldsymbol{x}+2\boldsymbol{e}_2)-c_1 \end{cases} \geq 0 \geq -\Delta_1 V(\boldsymbol{x}+\boldsymbol{e}_2)-c_1$ 时,

$$\Delta_{2,\boldsymbol{e}} T^1 V(\boldsymbol{x}) = \Delta_{2,\boldsymbol{e}} V(\boldsymbol{x}) + \Delta_1 V(\boldsymbol{x}+\boldsymbol{e}_2) + c_1 \geq 0$$

c) 当 $-\Delta_1 V(\boldsymbol{x}+\boldsymbol{e}_2-\boldsymbol{e}_1)-c_1 \geq 0 \geq -\Delta_1 V(\boldsymbol{x}+2\boldsymbol{e}_2)-c_1$ 时,

$$\Delta_{2,\boldsymbol{e}} T^1 V(\boldsymbol{x}) = \Delta_{2,2} V(\boldsymbol{x}) \geq 0$$

d) 当 $-\Delta_1 V(\boldsymbol{x}+\boldsymbol{e}_2-\boldsymbol{e}_1)-c_1 \geq 0 \geq -\Delta_1 V(\boldsymbol{x}-\boldsymbol{e}_1)-c_1$ 时,

$$\Delta_{2,\boldsymbol{e}} T^1 V(\boldsymbol{x}) = \Delta_{\boldsymbol{e},2} V(\boldsymbol{x}) + \Delta_{\boldsymbol{e},1} V(\boldsymbol{x}-\boldsymbol{e}_1) \geq 0$$

e) 当 $-\Delta_1 V(\boldsymbol{x}+\boldsymbol{e}_2-\boldsymbol{e}_1)-c_1 \geq 0 \geq \begin{cases} -\Delta_1 V(\boldsymbol{x}-\boldsymbol{e}_1)-c_1 \\ -\Delta_1 V(\boldsymbol{x}+2\boldsymbol{e}_2)-c_1 \end{cases}$ 时,

$$\begin{aligned}\Delta_{2,\boldsymbol{e}} T^1 V(\boldsymbol{x}) = {} & \Delta_{2,\boldsymbol{e}} V(\boldsymbol{x}-\boldsymbol{e}_1) - \\ & \Delta_1 V(\boldsymbol{x}+\boldsymbol{e}_2-\boldsymbol{e}_1) - c_1 \geq 0\end{aligned}$$

f) 当 $0 \geq -\Delta_1 V(\boldsymbol{x}+\boldsymbol{e}_2-\boldsymbol{e}_1)-c_1$ 时,

$$\Delta_{2,\boldsymbol{e}} T^1 V(\boldsymbol{x}) = \Delta_{2,\boldsymbol{e}} V(\boldsymbol{x}-\boldsymbol{e}_1) \geq 0$$

可知 $x_1=0$ 及 $x_1 \neq 0$ 时都有 $\Delta_{2,\boldsymbol{e}} T^1 V(\boldsymbol{x}) \geq 0$.

于是 $T^1 V(\boldsymbol{x})$ 满足性质 A1 ∼ A3, 即 $T^1 V(\boldsymbol{x}) \in \mathcal{V}$. 同理可证 $T^2 V(\boldsymbol{x}) \in \mathcal{V}$. 由于 $h(\boldsymbol{x})$ 是单调增的凸函数, 它自然满足性质 A1 ∼ A3, 而函数空间 $\mathcal{V}$ 对线性运算是封闭的. 综上所述, 当 $V \in \mathcal{V}$ 时, 有 $TV \in \mathcal{V}$. □

定义如下函数:

$$S_k(x_{-k}) = \min\{x_k \geq 0 | \Delta_k V(\boldsymbol{x}) \geq 0\}$$
$$R_k(x_{-k}) = \min\{x_k \geq 0 | \Delta_k V(\boldsymbol{x} - \boldsymbol{e}_k) \geq -c_k\}$$
$$R_3^k(x_{-k}) = \min\{x_k \geq 0 | \Delta_e V(\boldsymbol{x} - \boldsymbol{e}) \geq -c_3\}$$

其中, $k = 1, 2$, $x_{-k} = x_j$ $(k \neq j)$, 于是有引理 2 成立.

引理 2.

1) $S_k(x_{-k}) \leq S_k(x_{-k}+1), S_k(x_{-k}+1) \leq S_k(x_{-k})+1$;

2) $R_k(x_{-k}) \leq R_k(x_{-k}+1), R_k(x_{-k}+1) \leq R_k(x_{-k})+1$;

3) $R_3^k(x_{-k}+1) \leq R_3^k(x_{-k})$.

证明. 以 $k = 1$ 为例, 根据 A2 和 $S_1(x_2)$ 的定义知, 当 $\Delta_1 V(S_1(x_2+1), x_2) \geq \Delta_1 V(S_1(x_2+1), x_2+1) \geq 0$ 时, 可得: $\Delta_1 V(S_1(x_2+1), x_2) \geq 0$, 由 $S_1(x_2)$ 的定义知 $S_1(x_2) \leq S_1(x_2+1)$.

由 A3 知, 当 $\Delta_1 V(S_1(x_2)+1, x_2+1) \geq \Delta_1 V(S_1(x_2), x_2) \geq 0$ 时, 可得: $\Delta_1 V(S_1(x_2)+1, x_2+1) \geq 0$. 由 $S_1(x_2)$ 的定义知 $S_k(x_{-k}+1) \leq S_k(x_{-k})+1$, 同理可证 2) 成立.

由 A3 知, 当 $\Delta_{\boldsymbol{e}}(R_3^1(x_2), x_2+1) \geq \Delta_{\boldsymbol{e}}(R_3^1(x_2), x_2) \geq -c_3$ 时, $\Delta_{\boldsymbol{e}}(R_3^1(x_2), x_2+1) \geq -c_3$. 由 $R_3^1(x_2)$ 的定义知, $R_1(x_2+1) \leq R_1(x_2)$. □

基于引理 1 和引理 2, 有如下定理成立[13].

定理 1. 系统存在如下的控制策略, 当系统处于状态 (x_1, x_2) 时, 第 i 种组件的最优生产策略是动态的基库存策略, 即存在基库存水平 $S_i(x_{-i})$, 当且仅当 $x_i < S_i(x_{-i})$ 时需要生产组件 i. 组件的最优分配策略是动态的阈值策略, 即存在阈值水平 $R_i(x_{-i})$, 当 $x_i \geq R_i(x_{-i})$ 时满足第 i $(i = 1, 2)$ 类需求; 存在阈值水平 $(R_3^1(x_2), R_3^2(x_1))$, 当 $x_1 \geq R_3^1(x_2)$ 且 $x_2 \geq R_3^2(x_1)$ 时, 满足第 3 类需求, 否则拒绝. 最优控制策略还具有如下性质:

1) $S_i(x_{-i})$ 是 x_{-i} 的非降函数, 即随着 x_{-i} 的增大, 组件 i 的生产基库存水平要么增大要么不变, 但不会减小, 并且 x_{-i} 每增加 1, 组件 i 的基库存水平至多增加 1 个单位, 即 $S_i(x_{-i}+1) \leq S_i(x_{-i})+1$.

2) $R_i(x_{-i})$ 是 x_{-i} 的非降函数, 即随着 x_{-i} 的增大, 第 i 类需求的分配阈值要么增大要么不变, 但不会减小, 并且有 $R_i(x_{-i}+1) \leq R_i(x_{-i})+1$ $(i = 1, 2)$; 而 $R_3^j(x_{-j})$ 是 x_{-j} 的非增函数, 即随着 x_{-j} 的增大, 第 3 类需求组件 j 的分配阈值水平要么减小要么不变, 但不会增加.

图 2 和图 3 分别描述了系统的最优生产策略和库存分配策略的结构, 并且在图中标明了各个区域相应的最优行为策略.

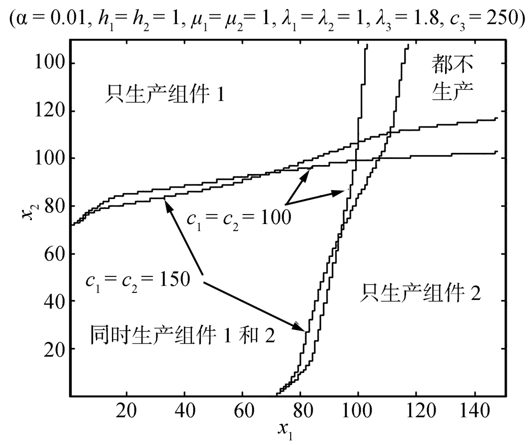

图 2 最优生产策略的结构

Fig. 2 The structure of the optimal production policy

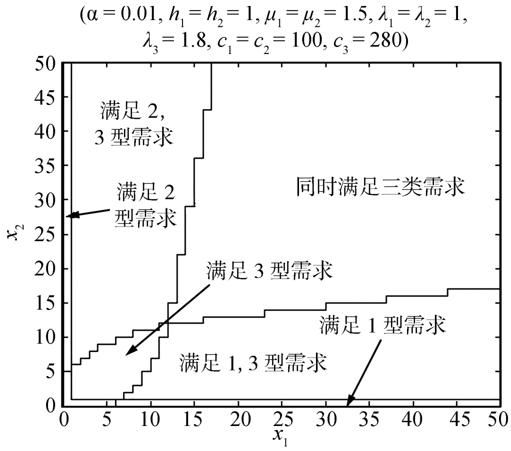

图 3 最优分配策略

Fig. 3 The structure of the optimal allocation policy

3 数值试验

为了进一步分析各参数对最优分配策略的影响, 我们进行了一系列的数值试验. 数值试验首先利用值迭代方法[14] 求解最优动态方程 (2) 得到最优成本函数在其状态空间上的取值. 算法迭代的状态空间截取在 $\{0, n_1^{\max}\} \times \{0, n_2^{\max}\}$, 其中 $n_1^{\max}$, $n_2^{\max}$ 的取值足够大, 使得最优成本函数的取值基本不再受截取水平影响. 值迭代过程连续两次迭代结果的误差精确到四位小数时迭代停止. 然后利用求得的最优成本函数 (记为 V) 得到最优的库存控制策略, 即当系统处于状态 $\boldsymbol{x}$ 时, 如果 $V(\boldsymbol{x}+\boldsymbol{e}_i) - V(\boldsymbol{x}) \leq 0$, 则应该开始生产该组件; 否则停止生产. 当有现货时, 如果 $V(\boldsymbol{x}) + c_i \geq V(\boldsymbol{x} - \boldsymbol{e}_i)$, 则满足第 i 类需求; 否则拒绝.

图 4 和图 5 分别讨论了 λ_1 和 λ_3 对最优分配策略的影响. 从图 4 可以看出, 其他条件不变, λ_1 越小, 则第 1 类需求的分配阈值越低, 第 2 类需求的分配阈值越高. 当组件 1 的需求率远远小于组件 2 的需求率, 并且 1 型需求的拒绝成本不是很低的情况下,

公司可以采取捆绑式销售策略. 在图 4 给出的最优控制策略中体现为, 当 λ_1 远远小于 λ_2 时, 第 1 类需求的分配阈值接近于 1, 而第 2 类需求的分配阈值较高. 只有当组件 2 的库存水平高于阈值水平时, 才满足第 2 类需求. 只要有现货, 尽量满足第 1, 3 类需求. 从图 5 中可以看出, 若其他条件不变, λ_3 越大, 则第 1, 2 类需求的分配阈值越高, 第 3 类需求的分配阈值越低. 这是因为第 3 类需求的到达率越大, 企业更倾向于将组件 1 和 2 留存下来, 满足未来到达的第 3 类顾客需求.

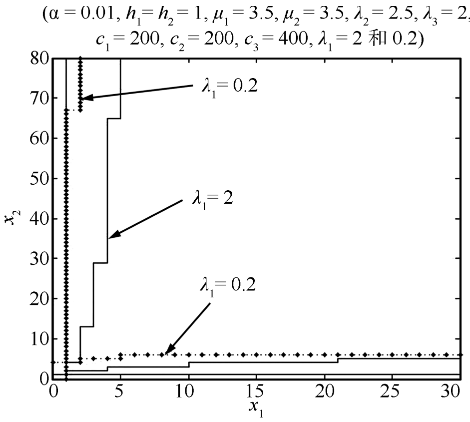

图 4　λ_1 对最优分配策略的影响

Fig. 4　The optimal allocation policy vs. λ_1 with $c_1 + c_2 < c_3$

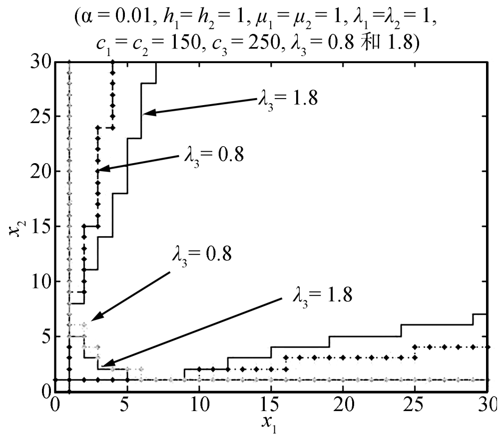

图 5　λ_3 对最优分配策略的影响

Fig. 5　The optimal allocation policy vs. λ_3 with $c_1 + c_2 > c_3$

图 6 和图 7 分别讨论了当 $c_1 + c_2 < c_3$ 时, c_1 和 c_3 对最优分配策略的影响. 从图 6 可以看出, 其他条件不变, c_1 越大, 则第 1, 2 类需求的分配阈值都越低, 此时降低第 1 类需求分配阈值的同时还要降低第 2 类需求的分配阈值是为了尽量让组件 1 和 2 的库存水平与第 1, 2 类顾客需求的到达率相一致. 从图 7 可以看出, 其他条件不变, c_3 越大, 则第 1, 2 类需求的分配阈值越高. 需要说明的是, 当 $c_1 + c_2 < c_3$ 时, 数值试验给出的结果是, 只要有现货, 第 3 类需求总是满足的, 遗憾的是, 理论上证明该性质存在困难.

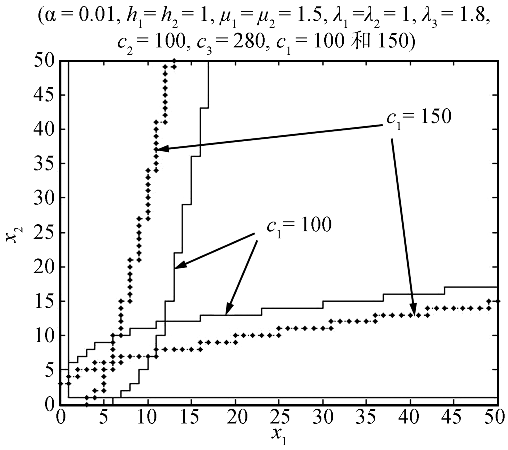

图 6　$c_1 + c_2 < c_3$ 时 c_1 对最优分配策略的影响

Fig. 6　The optimal allocation policy vs. c_1 with $c_1 + c_2 < c_3$

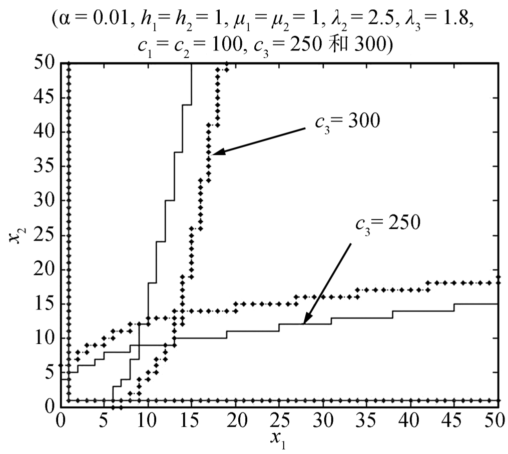

图 7　$c_1 + c_2 < c_3$ 时 c_3 对最优分配策略的影响

Fig. 7　The optimal allocation policy vs. c_3 with $c_1 + c_2 < c_3$

图 8 讨论了当 $c_1 + c_2 > c_3$ 时, c_1 对最优分配策略的影响. 其他条件不变, c_1 越大, 则第 1, 2 类需求的分配阈值越低, 第 3 类需求的分配阈值越高. c_3 变化对各类需求最优分配策略的影响反之.

4　小结与展望

针对由两种组件、三类顾客需求组成的按单装配系统, 本文研究了其中的组件生产控制与库存分配问题. 运用马尔科夫决策理论, 得到了最优控制方程, 得到了最优控制策略的结构性质, 证明了最优控制策略是依赖于系统状态的动态策略. 各组件最优生产策略是动态基库存策略, 其中基库存水平是其他组件库存水平的非减函数; 最优组件库存分配策

略是动态的阈值策略. 我们还用数值试验分析了各组参数对最优分配策略的影响, 为企业决策提供参考.

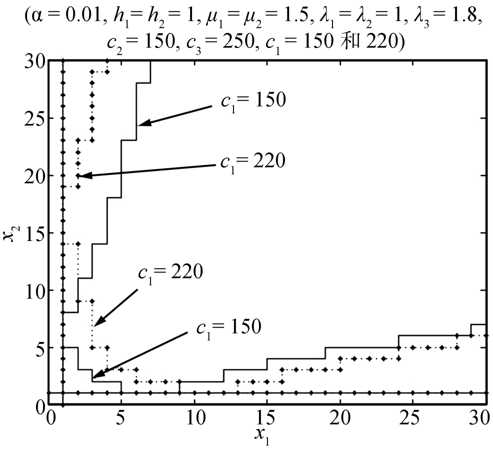

图 8 $c_1 + c_2 > c_3$ 时 c_1 对最优分配策略的影响

Fig. 8 The optimal allocation policy vs. c_1 with $c_1 + c_2 > c_3$

本文的研究工作可以在多方面进行推广, 比如部件生产由一台设备进行生产、生产设备存在故障、部件的生产时间服从一般的分布、各类需求到达过程为复合泊松过程等情况, 以及更加一般的多种组件、多类产品需求的按单装配系统.

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